VI.

Annotationes
ad
theoriam atque historiam perturbationum coelestium pertinentes

auctore

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§. I. 1) Nuper illustrissimus La Grange *), dum demonstrationem analyticam pararet propositionis a Poisson propositae: scilicet aequationem saecularem axis majoris planetarum non existere, si vel ad terminos formae $m'm'$ vel mm' respiciatur, vel ad secundam potentiam massarum (ut ajunt) — aequationes novas perturbatrices proposuit, quae et forma et simplicitate sunt memorabiles.

Nexum

*) Memoires de l'Institut Tome IX.

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Nexum harum aequationum atque istarum, quae hactenus ab astronomis usitatae fuerunt, hoc §pho ostendere conabor; opera, ut spero, non inutilis, cum et ad vulgares istas aequationes lux exinde aliqua redundet, simplicitati istarum proficua.

Demonstratio haec ex ipsis elementis perturbationum petita, formam supponet aequationum, quae et legibus quibusdam, quas perturbationes reciprocae sequuntur, favere videtur, casusque qui in mutua corporum coelestium relatione quoad situm planorum atque axium obtinere possent, simplici ratione complectitur.

2) Cum aequationes novae variationibus functionis cujusdam Ω, pendentibus a variatione elementorum, (scil. axis, eccentricitatis, nodi etc.) innitantur, natura atque mutationes hujus functionis ante omnia sunt explicandae.

Sumatur, (omisso ut in sequentibus factore qui a massa pendit)

$\Omega = \delta^{-\frac{1}{2}} - \frac{(P)}{\gamma'^3}$, dum δ mutuam planetarum perturbationibus affectorum distantiam, γ' radium vectorem planetae turbantis designet: obtinebitur

$$\frac{d\,\Omega}{dp} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\frac{d(P)}{dp} \qquad \left\{\begin{array}{l} p,\ \mathfrak{d},\ \varphi \text{ distantiam perihelii a nodo;} \\ \text{nodum; inclinationem ad planum fixum designantibus.} \end{array}\right\}$$

$$\frac{d\,\Omega}{d\mathfrak{d}} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\frac{d(P)}{d\mathfrak{d}} \qquad p',\ \mathfrak{d}',\ \varphi, \text{ eaedem quantitates relatae ad planetam turbantem.}$$

$$\frac{d\,\Omega}{d\varphi} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\frac{d(P)}{d\varphi}.$$

3) P quantitas ut ita dicam symmetrica est, a coordinatis (planetarum mutuo agentium) in orbita pendens, coefficientibusque, qui observationibus determinantur. Sint itaque x, y; x', y' coordinatae rectangulares in plano planetae turbati, atque turbantis; erit

(P)

$(P) = (A)\, x x' + (B)\, y y' + (C)\, x y' + (D)\, y x'$

$(A) = + Fa + Lb + Mc - Nd \qquad a = \cos.(\mathfrak{d}' - \mathfrak{d})$

$(B) = + Fb + La + Md - Nc \qquad b = \cos.(\mathfrak{d}' - \mathfrak{d}) \cos.\varphi' \cos.\varphi + \sin.\varphi' \sin.\varphi$

$(C) = - Lc + Mb - Na - Fd \qquad c = \sin.(\mathfrak{d}' - \mathfrak{d}) \cos.\varphi$

$(D) = + Ld - Ma + Nb + Fc \qquad d = \sin.(\mathfrak{d}' - \mathfrak{d}) \cos.\varphi'$

$F = \cos.p \cos.p'$; $L = \sin.p \sin.p'$; $M = \sin.p \cos.p'$; $N = \cos.p \sin.p'$.

Quantitas haec (P) oritur reducendo functionem symmetricam $XX' + YY' + ZZ'$ (in qua X, Y, Z.. designent coordinatas orthogonales ad planum fixum) ad coordinatas x, y..

Coefficientes (A), (B), (C), (D) variis reductionibus, varias inducere formas in aperto est.

4) Ex forma coefficientium 3) proposita confestim obtinetur

$$\frac{d(P)}{dp} = (D)\, x x' - (C)\, y y' + (B)\, x y' - (A)\, y x'$$

simulque proclivis est observatio: planum fixum, cum arbitrarium sit in dispositione generali, transire posse per punctum intersectionis orbitarum; hinc et $\mathfrak{d}' = \mathfrak{d}$ sumitur, et $c = d = 0$ evanescit. Itaque si ea sit mutua planetarum constitutio, quoad situm axium, ut $p = p' = 0$ sumi possit, aequatio formam hanc simplicem obtinebit

$$\frac{d(P)}{dp} = \cos.(\varphi' - \varphi)\, x y' - y x'.$$

5) Aequationes, quibus a, b, c, d determinantur, has suppeditant

$$\frac{da}{d\varphi} = 0;\ \frac{db}{d\varphi} = -\sin.\varphi \cos.(\mathfrak{d}' - \mathfrak{d}) \cos.\varphi' + \sin.\varphi' \cos.\varphi;$$

$$\frac{dc}{d\varphi} = -\sin.(\mathfrak{d}' - \mathfrak{d}) \sin.\varphi;\ \frac{d.d}{d\varphi} = 0.$$

Ex his conflatur aequatio

$$\begin{array}{rlrl}
\dfrac{d(P)}{d\varphi} = & xx'\sin.p\left\{+\dfrac{db}{d\varphi}\sin.p'+\dfrac{dc}{d\varphi}\cos.p'\right\} & = & xx'\left\{L\dfrac{db}{d\varphi}+M\dfrac{dc}{d\varphi}\right\}\\
 & yy'\cos.p\left\{+\dfrac{db}{d\varphi}\cos.p'-\dfrac{dc}{d\varphi}\sin.p'\right\} & & yy'\left\{F\dfrac{db}{d\varphi}-N\dfrac{dc}{d\varphi}\right\}\\
 & xy'\sin.p\left\{+\dfrac{db}{d\varphi}\cos.p'-\dfrac{dc}{d\varphi}\sin.p'\right\} & & xy'\left\{M\dfrac{db}{d\varphi}-L\dfrac{dc}{d\varphi}\right\}\\
 & yx'\cos.p\left\{+\dfrac{db}{d\varphi}\sin.p'+\dfrac{dc}{d\varphi}\cos.p'\right\} & & yx'\left\{N\dfrac{db}{d\varphi}+F\dfrac{dc}{d\varphi}\right\}
\end{array}$$

Observare licet (nro 4), sumi posse in disquisitione generali $\mathfrak{d}'=\mathfrak{d}$; hinc et $\frac{dc}{d\varphi}$ evanescit; $\frac{db}{d\varphi}$ obtinetur $=\sin.(\varphi'-\varphi)$; hinc si ea sit mutua planetarum constitutio ut sumi possit $p=p'=o$; aequatio aderit $\frac{d(P)}{d\varphi}=\sin.(\varphi'-\varphi)\,yy'$.

6) Denique adsunt aequationes ex 3)

$$\frac{da}{d\mathfrak{d}}=\sin.(\mathfrak{d}'-\mathfrak{d});\quad \frac{db}{d\mathfrak{d}}=\sin.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi'\cos.\varphi;\quad \frac{dc}{d\mathfrak{d}}=-\cos.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi;$$

$$\frac{dd}{d\mathfrak{d}}=-\cos.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi'$$

Ex quibus sponte fluunt sequentes inter $a, b, c, d,$ atque illarum variationes,

$$d.\cos.\varphi-\frac{db}{d\mathfrak{d}}=o;\qquad c\cos.\varphi-\frac{da}{d\mathfrak{d}}=\frac{dc}{d\varphi}.\sin.\varphi$$

$$a.\cos.\varphi+\frac{dc}{d\mathfrak{d}}=o\qquad b\cos.\varphi+\frac{dd}{d\mathfrak{d}}=\frac{db}{d\varphi}.\sin.\varphi.$$

Ex his aequationibus, differentiando aequationes 3), quibus natura quantitatis (P) determinatur, eruitur sequens, juncta aequatione nro. 4)

cos.

$$\cos.\varphi.\frac{d(P)}{dp}-\frac{d(P)}{d\mathfrak{v}}=xx'\left\{\begin{array}{l}+L\left\{d\cos.\varphi-\dfrac{db}{d\mathfrak{v}}\right\}\\-M\left\{a\cos.\varphi+\dfrac{dc}{d\mathfrak{v}}\right\}\\-N\left\{b\cos.\varphi+\dfrac{dd}{d\mathfrak{v}}\right\}\\+F\left\{c\cos.\varphi-\dfrac{da}{d\mathfrak{v}}\right\}\end{array}\right\}+yy'\left\{\begin{array}{l}-L\left\{c\cos.\varphi-\dfrac{da}{d\mathfrak{v}}\right\}\\+M\left\{b\cos.\varphi+\dfrac{dd}{d\mathfrak{v}}\right\}\\-N\left\{a\cos.\varphi+\dfrac{dc}{d\mathfrak{v}}\right\}\\-F\left\{d\cos.\varphi-\dfrac{db}{d\mathfrak{v}}\right\}\end{array}\right\}$$

$$+xy'\left\{\begin{array}{l}F\left\{b\cos.\varphi+\dfrac{dd}{d\mathfrak{v}}\right\}\\L\left\{a\cos.\varphi+\dfrac{dc}{d\mathfrak{v}}\right\}\\M\left\{d\cos.\varphi-\dfrac{db}{d\mathfrak{v}}\right\}\\N\left\{-c\cos.\varphi+\dfrac{da}{d\mathfrak{v}}\right\}\end{array}\right\}+yx'\left\{\begin{array}{l}F\left\{-a\cos.\varphi-\dfrac{dc}{d\mathfrak{v}}\right\}\\L\left\{-b\cos.\varphi-\dfrac{dd}{d\mathfrak{v}}\right\}\\M\left\{-c\cos.\varphi+\dfrac{da}{d\mathfrak{v}}\right\}\\N\left\{+d\cos.\varphi-\dfrac{db}{d\mathfrak{v}}\right\}\end{array}\right\}$$

scilicet,

$$\begin{aligned}\cos.\varphi.\frac{d(P)}{dp}-\frac{d(P)}{d\mathfrak{v}}=&+xx'\left\{N\frac{db}{d\varphi}+F\frac{dc}{d\varphi}\right\}\sin.\varphi= &&\sin.\varphi\, xx'\cos.p\left\{+\frac{db}{d\varphi}\sin.p'+\frac{dc}{d\varphi}\cos.p'\right\}\\&+yy'\left\{-M\frac{db}{d\varphi}+L\frac{dc}{d\varphi}\right\}\sin.\varphi &&+\sin.\varphi\, yy'\sin.p\left\{-\frac{db}{d\varphi}\cos.p'+\frac{dc}{d\varphi}\sin.p'\right\}\\&+xy'\left\{F\frac{db}{d\varphi}-N\frac{dc}{d\varphi}\right\}\sin.\varphi &&+\sin.\varphi\, xy'\cos.p\left\{+\frac{db}{d\varphi}\cos.p'-\frac{dc}{d\varphi}\sin.p'\right\}\\&+yx'\left\{-L\frac{db}{d\varphi}-M\frac{dc}{d\varphi}\right\}\sin.\varphi &&+\sin.\varphi\, yx'\sin.p\left\{-\frac{db}{d\varphi}\sin.p'-\frac{dc}{d\varphi}\cos.p'\right\}\end{aligned}$$

Ex qua aequatione apparet, coefficientes his terminis junctos eosdem plane esse quam eos qui in aequatione $\frac{dP}{d\varphi}$ occurrunt; mutatis signis termini

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secundi ac quarti; atque loco sin. p posito cos. p et vice versa; additoque factore sin. φ.

7) Quibus jam paratis ad aequationes ipsas transeamus. Aequatio *nova* pro parametro quam per g designamus, quam affert La Grange, haec est

$\frac{dg}{2\sqrt{g}} = \frac{d\Omega}{dp}$; itaque secundum ea quae hactenus tradita sunt

$$\text{I.}\quad \frac{dg}{2\sqrt{g}} = -\left\{\frac{1}{y'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{(D)\,xx' - (C)\,yy' + (B)\,xy' - (A)\,yx'\right\}$$

Aequatio hactenus *usitata* haec erat

$$dg = \left\{\frac{1}{y'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{(P)\frac{dy^2}{dt} - 2y^2.\frac{d(P)}{dt}\right\}$$

$$= \left\{\frac{1}{y'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} + \left\{(A)xx' + (B)yy' + (C)xy' + (D)yx'\right\}\left\{2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right\} \\ - \left\{(A)x'\frac{dx}{dt} + (B)y'\frac{dy}{dt} + (C)\,y'\frac{dx}{dt} + (D)\,x'\frac{dy}{dt}\right\}\left\{2x^2 + 2y^2\right\}\end{array}\right\}$$

Reductionibus adhibitis, oritur

$$2\left\{\frac{1}{y'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{y\frac{dx}{dt} - x\frac{dy}{dt}\right\}\left\{x\,(B)\,y' + x\,(D)\,x' - y(A)x' - y(C)y'\right\};$$

scilicet

$$= -2\sqrt{g}\left(\frac{1}{y'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right)\left((D)\,xx' - (C)\,yy' + (B)\,xy' - (A)\,yx'\right);$$ quae

est aequatio nova, cujus demonstrationem quaesivimus, identica praecedenti novae.

8) Aequatio *nova*, variationes parametri determinans haec est:

II. $\sqrt{g}\sin.\varphi.\frac{d\mathfrak{d}}{dt}=\frac{d\Omega}{d\varphi};$

$$=-\left(\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right)\frac{dP}{d\varphi}\text{ ex nro. 2)}$$

$$\left.\begin{aligned}=-xx'\sin.p.&\left(\frac{db}{d\varphi}\sin.p'+\frac{dc}{d\varphi}\cos.p'\right)\\ -yy'\cos.p&\left(\frac{db}{d\varphi}\cos.p'-\frac{dc}{d\varphi}\sin.p'\right)\\ -xy'\sin.p&\left(\frac{db}{d\varphi}\cos.p'-\frac{dc}{d\varphi}\sin.p'\right)\\ -yx'\cos.p&\left(\frac{db}{d\varphi}\sin.p'+\frac{dc}{d\varphi}\cos.p'\right)\end{aligned}\right\}\left(\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right)\text{ ex nro. 5).}$$

Aequatio *vulgaris*, variis sub formis proposita, si deducatur ex aequationibus in Mecanica coelesti *) pro tribus quantitatibus dc, dc', dc'' prolatis, ita se habet

$$-\sqrt{g}\sin.\varphi.\frac{d\mathfrak{d}}{dt}=\left(\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right)\left(x\sin.p+y\cos.p\right)$$
$$\left\{X'\sin.\mathfrak{d}\sin.\varphi-Y'\cos.\mathfrak{d}\sin.\varphi+Z'\cos\varphi\right\}$$

Factor hanc aequationem intrans, pendens a coordinatis orthogonalibus (ad planum fixum relatis) planetae turbantis X', Y', Z', reductione ad coordinatas in orbita facta, methodo usitata prodibit

$$+\sin.\mathfrak{d}\sin.\varphi\left(\left\{\cos.\mathfrak{d}'\cos.p'-\sin.\mathfrak{d}'\sin.p'\cos.\varphi'\right\}x'\right.$$
$$\left.-\left\{\cos.\mathfrak{d}'\sin.p'+\sin.\mathfrak{d}'\cos.p'\cos.\varphi'\right\}y'\right)$$

$-\cos.$

*) Libro II. Cap. VIII. §. 64.

$$-\cos.\mathfrak{d}\sin.\varphi\left(\begin{array}{l}\{\sin.\mathfrak{d}'\cos.p'+\cos.\mathfrak{d}'\sin.p'\cos.\varphi'\}\,x' \\ \qquad -\{\sin.\mathfrak{d}'\sin.p'-\cos.\mathfrak{d}'\cos.p'\cos.\varphi'\}\,y'\end{array}\right)$$

$$+\quad\cos.\varphi\left(\sin.p'\sin.\varphi'\,x'+\cos.p'\sin.\varphi'\,y'\right)$$

ex quibus formulis factor iste tandem obtinetur

$$\begin{array}{l}
-x'\cos.p'\sin.(\mathfrak{d}'-\mathfrak{d})\sin.\varphi \\
-x'\sin.p'\{\cos.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi'\sin.\varphi-\sin.\varphi'\cos.\varphi\} \\
+y'\sin.p'\ \sin.(\mathfrak{d}'-\mathfrak{d})\sin.\varphi \\
-y'\cos.p'\{\cos.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi'\sin.\varphi-\sin.\varphi'\cos.\varphi\}
\end{array}$$

cujus coefficientes congruunt cum $\frac{dc}{d\varphi}$ et $\frac{db}{d\varphi}$ nro. 5).

Exinde prodit aequatio

$$-\sqrt{g}\sin.\varphi.d\mathfrak{d}=\left(\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right)\{x\sin.p+y\cos.p\}\left\{\begin{array}{l}+\left(\sin.p'\frac{db}{d\varphi}+\cos.p'\frac{dc}{d\varphi}\right)x' \\ +\left(\cos.p'\frac{db}{d\varphi}-\sin.p'\frac{dc}{d\varphi}\right)x'\end{array}\right\}$$

quae plane congruit cum aequatione *nova*, cujus demonstrationem paravimus.

9) Aequatio *nova* variationes Inclinationis determinans, a La Grange proposita, haec est

$$\sqrt{g}\,.\sin.\varphi\,.\frac{d\varphi}{dt}=\cos.\varphi\,.\frac{d\Omega}{dp}-\frac{d\Omega}{d\mathfrak{d}}$$

Aequatio haec, adhibitis reductionibus nro. 2) et 6), in hanc abit

$$\text{III. } \sqrt{g}\sin\varphi\frac{d\varphi}{dt}=-\left\{\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right\}\left(\begin{array}{l}+xx'\cos.p\left\{+\frac{db}{d\varphi}\sin.p'+\frac{dc}{d\varphi}\cos.p'\right\}\\ +yy'\sin.p\left\{-\frac{db}{d\varphi}\cos p'+\frac{dc}{d\varphi}\sin.p'\right\}\\ +xy'\cos.p\left\{+\frac{db}{d\varphi}\cos p'-\frac{dc}{d\varphi}\sin.p'\right\}\\ +yx'\sin.p\left\{-\frac{db}{d\varphi}\sin.p'-\frac{dc}{d\varphi}\cos.p'\right\}\end{array}\right)\sin.\varphi$$

Aequatio *usitata* prorsus et demonstratione et forma similis aequationi, variationes Nodi determinanti, haec est:

$$-\sqrt{g}.\frac{d\varphi}{dt}=\left\{\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right\}(x\cos p-y\sin p)(X'\sin\mathfrak{d}\sin p-Y'\cos\mathfrak{d}\sin\varphi+Z'\cos.\varphi)$$

quae reductionibus nro. 8) adhibitis in hanc abit

$$-\sqrt{g}.\frac{d\varphi}{dt}=\left\{\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right\}(x\cos.p-y\sin.p)\left(\begin{array}{l}+\left\{\sin.p'\frac{db}{d\varphi}+\cos.p'\frac{dc}{d\varphi}\right\}x'\\ +\left\{\cos.p'\frac{db}{d\varphi}-\sin.p'\frac{dc}{d\varphi}\right\}y'\end{array}\right)$$

Quam plane identicam esse cum aequatione *nova*, sponte apparet.

10) Demonstratum jam est, aequationes *novas*, quibus variationes parametri g, Nodi $\mathfrak{d}$, Inclinationis φ determinantur I, II, III, facili negotio derivari ab aequationibus vulgo notis. His additur aequatio pro variatione axis magni, quam a variatione functionis Ω pendere olim ab illustrissimo la Grange ostensum fuit: ita ut unica tantum aequatio supersit:

$$-\cos.\mathfrak{d}\sin.\varphi\left(\begin{array}{l}\{\sin.\mathfrak{d}'\cos.p'+\cos.\mathfrak{d}'\sin.p'\cos.\varphi'\}\,x'\\ \qquad\qquad -\{\sin.\mathfrak{d}'\sin.p'-\cos.\mathfrak{d}'\cos.p'\cos.\varphi'\}\,y'\end{array}\right)$$

$$+\qquad\cos.\varphi\left(\sin.p'\sin.\varphi'\,x'+\cos.p'\sin.\varphi'\,y'\right)$$

ex quibus formulis factor iste tandem obtinetur

$$\begin{array}{l}-x'\cos.p'\sin.(\mathfrak{d}'-\mathfrak{d})\sin.\varphi\\ -x'\sin.p'\{\cos.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi'\sin.\varphi-\sin.\varphi'\cos.\varphi\}\\ +y'\sin.p'\sin.(\mathfrak{d}'-\mathfrak{d})\sin.\varphi\\ -y'\cos.p'\{\cos.(\mathfrak{d}'-\mathfrak{d})\cos.\varphi'\sin.\varphi-\sin.\varphi'\cos.\varphi\}\end{array}$$

cujus coefficientes congruunt cum $\frac{dc}{d\varphi}$ et $\frac{db}{d\varphi}$ nro. 5).

Exinde prodit aequatio

$$-\sqrt{g}\sin.\varphi.d\mathfrak{d}=\left(\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right)\{x\sin.p+y\cos.p\}\left\{\begin{array}{l}+\left(\sin.p'\frac{db}{d\varphi}+\cos.p'\frac{dc}{d\varphi}\right)x'\\ +\left(\cos.p'\frac{db}{d\varphi}-\sin.p'\frac{dc}{d\varphi}\right)x'\end{array}\right\}$$

quae plane congruit cum aequatione *nova*, cujus demonstrationem paravimus.

9) Aequatio *nova* variationes Inclinationis determinans, a La Grange proposita, haec est

$$\sqrt{g}.\sin.\varphi.\frac{d\varphi}{dt}=\cos.\varphi.\frac{d\Omega}{dp}-\frac{d\Omega}{d\mathfrak{d}}$$

Aequatio haec, adhibitis reductionibus nro. 2) et 6), in hanc abit

$$\text{III. } \sqrt{g}\sin\varphi\frac{d\varphi}{dt}=-\left\{\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right\}\left[\begin{array}{l}+xx'\cos.p\left\{+\frac{db}{d\varphi}\sin.p'+\frac{dc}{d\varphi}\cos.p'\right\}\\[2ex] +yy'\sin.p\left\{-\frac{db}{d\varphi}\cos p'+\frac{dc}{d\varphi}\sin.p'\right\}\\[2ex] +xy'\cos.p\left\{+\frac{db}{d\varphi}\cos p'-\frac{dc}{d\varphi}\sin.p'\right\}\\[2ex] +yx'\sin.p\left\{-\frac{db}{d\varphi}\sin.p'-\frac{dc}{d\varphi}\cos.p'\right\}\end{array}\right]\sin.\varphi$$

Aequatio *usitata* prorsus et demonstratione et forma similis aequationi, variationes Nodi determinanti, haec est:

$$-\sqrt{g}.\frac{d\varphi}{dt}=\left\{\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right\}(x\cos p-y\sin p)(X'\sin\mathfrak{d}\sin p-Y'\cos\mathfrak{d}\sin\varphi+Z'\cos.\varphi)$$

quae reductionibus nro. 8) adhibitis in hanc abit

$$-\sqrt{g}\cdot\frac{d\varphi}{dt}=\left\{\frac{1}{\gamma'^3}-\frac{1}{\delta^{\frac{3}{2}}}\right\}(x\cos.p-y\sin.p)\left[\begin{array}{l}+\left\{\sin.p'\frac{db}{d\varphi}+\cos.p'\frac{dc}{d\varphi}\right\}x'\\[2ex] +\left\{\cos.p'\frac{db}{d\varphi}-\sin.p'\frac{dc}{d\varphi}\right\}y'\end{array}\right]$$

Quam plane identicam esse cum aequatione *nova*, sponte apparet.

10) Demonstratum jam est, aequationes *novas*, quibus variationes parametri g, Nodi $\mathfrak{d}$, Inclinationis φ determinantur I, II, III, facili negotio derivari ab aequationibus vulgo notis. His additur aequatio pro variatione axis magni, quam a variatione functionis Ω pendere olim ab illustrissimo la Grange ostensum fuit: ita ut unica tantum aequatio supersit:

 11)

11) Antequam ad hanc probandam transeamus, adnotationes quasdam, quas forma singularis aequationum propositarum postulare videtur, hic proponemus. Positis $\mathfrak{d}' = \mathfrak{d}$, quae hypothesis semper locum habet, cum situs plani fixi sit arbitrarius, *casu* quo duorum planetarum se turbantium ea erit constitutio ut $p' = p = o$ poni queat, aderunt aequationes

$$\frac{1}{2\sqrt{g}} \cdot \frac{dg}{dt} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\} \left\{\sin.(\varphi' - \varphi)\, xy' - yx'\right\}$$

$$\sqrt{g} \cdot \sin.\varphi \cdot \frac{d\mathfrak{d}}{dt} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\} \sin.(\varphi' - \varphi)\, yy'$$

$$\sqrt{g} \cdot \frac{d\varphi}{dt} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\} \sin.(\varphi' - \varphi)\, xy'$$

12) Si variationes mutuae duorum planetarum considerentur, forma aequationum hactenus tractatarum symmetricam quandam prae se fert speciem, si ad quantitates xx', yy', xy', yx' respicias. Coefficientes solummodo, quibus hae quantitates affectae sunt, diversi sunt; factorque $\left(\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right)$, si de perturbationibus reciprocis quaestio est, abit in $\left(\frac{1}{\gamma^3} - \frac{1}{\delta^{\frac{3}{2}}}\right)$; ita ut in his *perturbationibus mutuis infinita occurrat terminorum multitudo, qui in ratione constanti sunt,* scilicet in ratione horum coefficientium.

13) Simili ratione apparet perturbationes unius ejusdem planetae quoad parametrum g; Nodum $\mathfrak{d}$; inclinationem φ continere multitudinem membrorum, quae coefficientibus tantum differant; ita ut calculi numerici explicatio solummodo sola quantitate $\delta^{\frac{3}{2}}$ intrica-

tior

tior fiat; qua explanata reliquae sint satis expeditae calculi partes. Haec annotatio *ipsam calculi praxin* adjuvare potest.

14) Hanc disquisitionem exemplis illustrare, commodum erit. Pallas et Juno actione atque attractione mutua se petentes in calculum vocentur. γ designet radium vectorem Palladis; γ' radium vectorem Junonis; m, m' massas (ut ajunt) planetarum; tres aequationes, de quibus hoc §pho sermo erat, ita se habebunt (x, y; x', y' denotant ut supra coordinatas orthogonales in orbita).

Aequationes determinantes variationes parametri etc. actione Palladis et Junonis reciproca oriundae.

Aequationes pro Pallade, turbata a Junone;

I. $$\frac{dg}{dt} = -m'\,2\sqrt{g}\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} -0{,}88561\;xx' \\ -0{,}90322\;yy' \\ +0{,}39618\;xy \\ -0{,}32847\;yx' \end{array}\right\}$$

II. $$\frac{d\mathfrak{d}}{dt} = -\frac{m'}{\sqrt{g}.\sin\varphi}\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} -0{,}24767\;xx' \\ +0{,}11607\;yy' \\ -0{,}14548\;xy' \\ +0{,}19760\;yx' \end{array}\right\}$$

III. $$\frac{d\varphi}{dt} = -\frac{m'}{\sqrt{g}}\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} +0{,}19760\;xx' \\ +0{,}14548\;yy' \\ +0{,}11607\;xy' \\ +0{,}24767\;yx' \end{array}\right\}$$

Aequationes pro Junone turbata a Pallade.

$$\frac{dg'}{dt} = -m\,2\sqrt{g'}\left\{\frac{1}{\gamma^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} +0{,}90322\;xx' \\ +0{,}88561\;yy' \\ -0{,}32847\;xy' \\ +0{,}39618\;yx' \end{array}\right\}$$

$$\frac{d\mathfrak{d}'}{dt} = -\frac{m}{\sqrt{g'}\sin\varphi'}\left\{\frac{1}{\gamma^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} +0{,}25794\;xx' \\ -0{,}10420\;yy' \\ +0{,}13571\;xy' \\ -0{,}19804\;yx' \end{array}\right\}$$

$$\frac{d\varphi'}{dt} = -\frac{m}{\sqrt{g}}\left\{\frac{1}{\gamma^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\}\left\{\begin{array}{l} +0{,}13571\;xx' \\ +0{,}19804\;yy' \\ -0{,}10420\;xy' \\ -0{,}25794\;yx' \end{array}\right\}$$

15) Aequationes differentiales allatae primi gradus, in quibus t denotat tempus, integrationem directam admittunt; quando $\delta^{-\frac{3}{2}}$

 expli-

explicari potest per terminos cosinus aut sinus motus medii continentes: quod in systemate planetarum semper locum habere, demonstrandum erit in sequentibus. Quantitates x, y; x', y' similiter ita explicari atque evolvi, notum est.

16) Restat jam aequatio, qua variationes perihelii determinantur.

Jam formulae sequentes ex theoria motus elliptici sine negotio derivantur

$$\gamma \cdot \frac{d\gamma}{de} = -ax; \; \frac{dx}{de} = -\frac{a^2 \sin. u^2}{\gamma} - a = \frac{y \frac{dx}{dt} - a}{na\sqrt{1-e^2}}$$

$$\gamma \cdot \frac{d\gamma}{dt} = \frac{e}{\sqrt{g}} y; \; \frac{dy}{de} = -\frac{ae \sin. u}{\sqrt{1-e^2}} + a^2 \sqrt{1-e^2} \sin u \cos u = -\frac{e y}{1-e^2} + \frac{y \frac{dy}{dt}}{na\sqrt{1-e^2}}$$

Ex his, substitutionibus factis, obtinetur

$$\frac{d\Omega}{de} = -\left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\} \frac{y}{na\sqrt{1-e^2}} \left\{(A)x' \frac{dx}{dt} + (B)y' \frac{dy}{dt} + (C)y' \frac{dx}{dt} + (D)x' \frac{dy}{dt}\right\}$$
$$+ \left\{\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}\right\} \left\{a(A)x' + a(C)y' + \frac{e}{1-e^2}(B)yy' + \frac{e}{1-e^2}(D)yx'\right\}$$
$$+ \quad \delta^{-\frac{3}{2}} ax$$

Jam si ex aequationibus hactenus *usitatis*, quas offert Mechan. celest. *) ex aequationibus pro quantitatibus df, df', df'' deducatur aequatio variationes perihelii exhibens, obtinetur (introducendo coefficientes α, β;)

*) Lib. II. Cap. VIII. §. 64.

$$\frac{dp}{dt} + \cos.\varphi \frac{d\mathfrak{d}}{dt} = {}^{*)} R \left\{ \frac{(P)\frac{dy}{dt} - 2y\frac{d(P)}{dt}}{e} + \left\{ \frac{\alpha Y' - \beta X'}{\cos.\varphi \,.\, e} \right\} . \gamma \frac{d\gamma}{dt} \right\}$$

$$- \frac{\cos.p \sin.\varphi}{\cos.\varphi} \left(\cos.p \sin.\varphi \frac{d\mathfrak{d}}{dt} - \sin.p . \frac{d\varphi}{dt} \right)$$

$$- \delta^{-\frac{3}{2}} . \left(\gamma \frac{d\gamma}{dt} y - \gamma^2 \frac{dy}{dt} \right) \frac{1}{e}$$

Ut haec aequatio analoga reddatur praecedenti, notandae sunt formulae sequentes, quarum demonstratio obvia; scilicet

$$(P)\frac{dy}{dt} - y\frac{d(P)}{dt} = \sqrt{g} \left((A)x' + (C)y' \right);$$ (A) et (C) nro. 3 determinatae;

$$\gamma \frac{d\gamma}{dt} . y - \gamma^2 \frac{dy}{dt} = - x\sqrt{g}$$

Similiter ex aequationibus II et III nro. 8 et 9. sequitur

$$\cos.p \sin.\varphi \frac{d\mathfrak{d}}{dt} - \sin.p \frac{d\varphi}{dt} = - m' \left(\frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}} \right) \left(\begin{array}{l} yy' \left(\cos p \left\{ \frac{db}{d\varphi} \cos.p' - \frac{dc}{d\varphi} \sin.p' \right\} \right) \\ yx' \left(\sin.p \left\{ \frac{db}{d\varphi} \sin.p' + \frac{dc}{d\varphi} \cos p' \right\} \right) \end{array} \right)$$

Porro cum coefficientes α, β his aequationibus determinentur

$$\alpha = \cos.\mathfrak{d} \cos.p - \sin.\mathfrak{d} \sin.p \cos.\varphi$$
$$\beta = \sin.\mathfrak{d} \cos.p + \cos.\mathfrak{d} \sin.p \cos.\varphi$$

atque valores Y', X' ope coefficientium, qui quantitatibus α, β analogi sunt ad coordinatas y', x' reduci queant; colligentur aequationis termini productis yy' atque yx' juncti, ope reductionum, quas offerunt aequationes $\alpha \cos.\mathfrak{d}' - \alpha \sin.\mathfrak{d}' - \cos.p \sin.\varphi . \frac{dc}{d\varphi}$ atque

$$(\beta \sin.\mathfrak{d}' + \alpha \cos.\mathfrak{d}') \cos.\varphi' + \cos.p \sin.\varphi \frac{db}{d\varphi}.$$

Quo

*) $R = \frac{1}{\gamma'^3} - \frac{1}{\delta^{\frac{3}{2}}}$.

Quo facto identitas coefficientium, producta yy' yx' comitantium in aequationibus tum nova tum prius usitata sponte apparebit: (cujus evolutionem solummodo brevitatis gratia omittimus).

17) Caeterum simili ratione qua termini e membris $\left(\frac{\alpha . Y' - \beta X'}{\cos.\varphi\, e}\right) \frac{\gamma d\gamma}{dt}$ atque $\frac{\cos.p \sin.\varphi}{\cos.\varphi}\left(\cos.p \sin.\varphi \frac{d\mathfrak{d}}{dt} - \sin.p \frac{d\varphi}{dt}\right)$ reductionibus coalescunt, eadem quantitates $y \frac{d(P)}{dt}$ atque termini reductionibus superioribus oriundi coalescunt, ita ut aequatio evadat simplex satis

$$\text{IV.}\quad e\left(\frac{dp}{dt} + \cos\varphi \frac{d\mathfrak{d}}{dt}\right) = \frac{1}{\gamma'^3 - \delta^{\frac{3}{2}}}\left(+\frac{dx}{dt} \cdot \frac{d(P)}{dp} + \sqrt{g}\left\{(A)x' + (C)y'\right\}\right) + \delta^{-\frac{3}{2}} x \sqrt{g}$$

quam inter aequationes perturbatrices recipere nec usus practicus vetabit.

18) Haec aequatio differt forma ab aequationibus I. II. III. verum demonstrari potest aequationem, qua $\frac{de}{dt}$ (variationes eccentricitatis) determinantur simili plane forma gaudere: ita ut symmetria quaedam hac ratione restituatur.

VII.