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On Hermitian Block Toeplitz Matrices and Generalizations of a Theorem of C. Carathéodory

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1. Introduction

In this note, we are concerned with Hermitian block Toeplitz matrices, i.e. matrices of the form

$$T_n = T_n(C_0, C_1, \dots, C_n) := \begin{pmatrix} C_0 & C_1 & \dots & \dots & C_n \\ C_1^H & C_0 & C_1 & & \vdots \\ \vdots & C_1^H & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & C_1 \\ C_n^H & \dots & \dots & C_1^H & C_0 \end{pmatrix} \tag{1}$$

where $C_0, C_1, \dots, C_n$ are complex $q \times q$ matrices and $C_0 = C_0^H$. In the scalar case $q = 1$, these matrices were introduced by O. Toeplitz [23] in connection with the class of analytic functions

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad |z| < 1, \tag{2}$$

with

$$\operatorname{Re} f(z) \geq 0 \qquad \text{for all } |z| < 1. \tag{3}$$

These functions were first studied by C. Carathéodory [4,5] and are nowadays called Carathéodory functions. The classical Carathéodory-Toeplitz theorem states that an analytic function (2) satisfies (3) iff the Toeplitz matrices $T_n(c_0 + \bar{c}_0, c_1, c_2, \dots c_n)$ are positive semidefinite for $n = 0, 1, \dots$.

Investigations on the coefficients of functions (2), (3) led Carathéodory to the following

Theorem A. *Let $c_1, \ldots, c_n$ be given complex numbers not all zero, $n \geq 1$. There exists a minimal integer t and unique numbers $\varrho_j > 0$, $0 \leq \Phi_j < 2\pi$, $j = 1, \ldots t$, such that*

$$c_k = \sum_{j=1}^{t} \varrho_j e^{i\Phi_j k}, \qquad k = 1, \ldots, n. \tag{4}$$

Later on, a number of different proofs of this theorem appeared in the literature (Szegö [21, 22], Akhiezer and Krein [2, p. 24], Cybenko [8], Constantinescu [6]) reflecting the intimate connections between Carathéodory functions and orthogonal polynomials on the unit circle [16], the trigonometric moment problem [1] and positive semi-definite Toeplitz matrices. Recently, an application of Theorem A in signal processing [20] has stimulated some interest in computing the Carathéodory representation (4) numerically [9]. In this context, the proof given in [8] is important, since it gives a means to obtain (4) via solving the eigenvalue problem for a unitary matrix. This approach is based on the fact that any positive semi-definite Toeplitz matrix $T_n(c_0, c_1, \ldots c_n)$, $c_0 \in \boldsymbol{R}$, $c_1, \ldots, c_n \in \boldsymbol{C}$, can be written in the form

$$T_n = (b, U b, \ldots, U^n b)^H (b, U b, \ldots, U^n b). \tag{5}$$

Here $b \in \boldsymbol{C}^t$, $t = \mathit{rank}\ T_n$ and U is a unitary $t \times t$ matrix. In particular, given $c_1, \ldots c_n$, if we choose c_0 such that $T_n(c_0, \ldots, c_n)$ is positive semi-definite, but singular, the eigenvalues of U are just the numbers $e^{i\Phi_j}$ in (4).

In this note, we are mainly concerned with factorizations similar to (5) for arbitrary Hermitian block Toeplitz matrices and with generalizations of Theorem A. It turns out that an appropriately generalized factorization is not always possible, and a necessary and sufficient criterion for its existence is presented. Furthermore, we show that the factorization problem for (1) is equivalent to the singular extension problem for T_n and to the existence of T_n-unitary matrices of Frobenius type. A description of all solutions of these three problems is also given.

Our main results are stated in Sec. 3 and proved in Sec. 4. The proofs are essentially based on recent results on an extension problem for H-unitary matrices which are recalled in Sec. 2. Here and in the sequel, for a given Hermitian matrix H, a matrix U is called H-

unitary, if $U^H HU = H$. Finally, in Sec. 5 we deduce generalizations of Carathéodory's Theorem A.

In the last years, extensions of the classical Carathéodory-Toeplitz theorem were given which connect certain meromorphic and/or matrix-valued functions with classes of infinite Hermitian block Toeplitz matrices $T_\infty = T_\infty(C_0, C_1, \ldots)$ (see Krein and Langer [17, 18, 19], Delsarte, Genin and Kamp [11, 12] and the references quoted therein). Based on this link, methods of complex analysis and operator theory in infinite-dimensional spaces lead to results on the structure of infinite Toeplitz matrices T_∞. In contrast to this approach, we are dealing exclusively with finite block Toeplitz matrices, and all our results are obtained using only elementary matrix analysis.

2. An extension problem for H-unitary matrices

Let H be an Hermitian $n \times n$ matrix, and consider the following extension problem (P): If U_0 is a given $n \times m$ matrix $(0 \leq m < n)$ such that

$$U_0^H H U_0 = A \tag{6}$$

where A is the $m \times m$ leading principal submatrix of H, can U_0 be extended to an H-unitary matrix $U = (U_0\ U_1)$? Note that for $m = 0$, U_0 and A are empty matrices and (P) reduces to the problem of finding H-unitary matrices.

A complete solution of (P) was given in [13]. The result can be summarized as follows. There exists a nonsingular $n \times n$ matrix

$$S = \begin{array}{c} \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{matrix} \}m \\ \end{matrix} \\ \begin{matrix} \underbrace{\phantom{S_{11}}}_{m} & \phantom{S_{12}} \end{matrix} \end{array} \tag{7}$$

such that

$$S^H H S = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 & 0 \\ \hline 0 & 0 & I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} \}d_0 \\ \}m_0 \\ \}k \\ \\ \}n_0 - m_0 - 2k =: l \\ \}d_1 \end{matrix} \tag{8}$$

where Λ_1, Λ_2 are real signature matrices, i. e. diagonal matrices with diagonal elements $\pm$ 1, and

$$m_0 = rank\ A, \quad n_0 = rank\ H, \quad d_0 + m_0 + k = m.$$

We denote by $\tilde{H}$ the nonsingular part of the matrix (8) obtained by deleting the first d_0 and the last d_1 rows and columns. Furthermore, set

$$S^{-1} U_0 S_{11} = \begin{array}{cc} \left(\begin{array}{cc} Z_1 & Z_2 \\ \tilde{V}_0 & \tilde{U}_0 \\ Z_3 & Z_4 \end{array}\right) & \begin{array}{l} \}d_0 \\ \}n_0. \\ \}d_1 \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{d_0} & \underbrace{\qquad}_{m_0+k} \end{array} & \end{array} \tag{9}$$

Theorem B ([13, Thm. 2]). *The extension problem (P) has a solution iff*

$$\tilde{V}_0 = 0 \text{ and } rank\ \tilde{U}_0 = m_0 + k. \tag{10}$$

Moreover, if (10) holds, there exists an $\tilde{H}$-unitary matrix $L = (\tilde{U}_0\ \star)$ and the set of all H-unitary extensions of U_0 is given by

$$U = (U_0\ U_1) = S \left(\begin{array}{cc|cc} Z_1 & Z_2 & Z_5 & Z_6 \\ 0 & \tilde{U}_0 & LM & 0 \\ \underbrace{Z_3}_{d_0} & Z_4 & Z_7 & \underbrace{Z_8}_{d_1} \end{array}\right) S^{-1} \tag{11}$$

where

$$M = \begin{array}{cc} \left(\begin{array}{cc} 0 & 0 \\ N - \frac{1}{2} Y^H \Lambda_2 Y - & Y^H \Lambda_2 X \\ I & 0 \\ \underbrace{Y}_{k} & \underbrace{X}_{l} \end{array}\right) & \begin{array}{l} \}m_0 \\ \}k \\ \}k \\ \}l \end{array} \end{array}$$

and N, X, Y, Z_j, $j = 5, \ldots, 8$, are arbitrary complex matrices of appropriate dimension with $N = -N^H$ and X Λ_2-unitary.

Remarks 1. In the case $m = 0$, $\tilde{V}_0$ and $\tilde{U}_0$ are empty matrices, $m_0 = k = 0$, and thus (10) is always satisfied. Moreover, one can choose $L = I$ in (11).

2. (P) is solvable, if H is positive semi-definite (13, Corollary 1).

In the rest of this paper, we consider problem (P) exclusively for the special case that $H = T_n = T_n\ (C_0, \ldots, C_n)$ is an Hermitian block

Toeplitz matrix, $A = T_{n-1}(C_0, \ldots, C_{n-1})$ ($T_{-1} :=$ empty matrix, $rank\ T_{-1} := 0$), and U_0 is the shift matrix

$$U_0 := \underbrace{\begin{pmatrix} 0 \\ I \end{pmatrix}}_{nq} \begin{matrix} \}q \\ \}nq \end{matrix} . \tag{12}$$

The block Toeplitz structure of T_n yields $U_0^H T_n U_0 = T_{n-1}$, and thus (6) is satisfied. Two solutions $U = (U_0\ U_1)$ and $U' = (U_0\ U'_1)$ of (P) are said to be equivalent, if in their canonical representation (11) $M = M'$. The corresponding set of equivalence classes is denoted by $\mathcal{U}(T_n)$, and we set $\mathcal{U}(T_n) = \emptyset$, if (P) is not solvable.

3. Statement of the main results

Suppose that we are given integers $r, q, n \geq 1$, an $r \times r$ signature matrix

$$\Lambda = \begin{pmatrix} I_\mu & 0 \\ 0 & -I_\nu \end{pmatrix}, \ \mu + \nu = r, \tag{13}$$

a Λ-unitary matrix W, and an $r \times q$ matrix X such that the block Krylov matrix

$$K_n(X, W) := (X,\ WX,\ W^2X, \ldots, W^nX)$$

has rank r. Obviously,

$$T_n = K_n(X, W)^H \Lambda K_n(X, W) \tag{14}$$

is a matrix of type (1) with $C_k = X^H \Lambda W^k X$, $k = 0, \ldots, n$, and $rank\ T_n = r$. For $q = 1$ and $\Lambda = I$, (14) reduces to (5) and, as mentioned in the introduction, gives a representation of all positive semi-definite Toeplitz matrices. Therefore, it is natural to ask whether any arbitrary Hermitian block Toeplitz matrix admits a factorization (14).

From now on it is assumed that $T_n = T_n(C_0, \ldots, C_n)$ is an Hermitian matrix of the form (1) with $n \geq 0$ and block size $q \geq 1$, and we set $r = rank\ T_n$. The trivial case $T_n = 0$ is always excluded. We seek factorizations (14) with X $r \times q$ and W an $r \times r$ Λ-unitary matrix such that

$$rank\ K_n(X, W) = r. \tag{15}$$

By Sylvester's law of inertia, it follows from (14) and (15) that the matrix Λ in (13) is uniquely determined with μ and v being the number of positive and negative eigenvalues of T_n, respectively. Two factorizations

$$T_n = K_n(X_j, W_j)^H \Lambda K_n(X_j, W_j), j = 1, 2,$$

are said to be equivalent, if

$$X_2 = ZX_1,\ W_2 = ZW_1Z^{-1} \tag{16}$$

with Z a Λ-unitary matrix. By $F(T_n)$ we denote the set of equivalence classes of factorizations (14) and set $F(T_n) = \emptyset$, if no such representation exists.

By choosing any $q \times q$ matrix C_{n+1}, one obtains an extension $T_{n+1}(C_0, \ldots, C_n, C_{n+1})$ of $T_n(C_0, \ldots, C_n)$. T_{n+1} is called a singular extension of T_n, if

$$rank\ T_{n+1} = rank\ T_n. \tag{r_{n+1}}$$

We denote by $C(T_n)$ the set of all C_{n+1} which yield a singular extension of T_n.

The partitions

$$T_n = \begin{pmatrix} T_{n-1} & B_n \\ B_n^H & C_0 \end{pmatrix} = \begin{pmatrix} C_0 & D_n^H \\ D_n & T_{n-1} \end{pmatrix} \tag{17}$$

with

$$B_n = \begin{pmatrix} C_n \\ C_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ C_1 \end{pmatrix} \text{ and } D_n = \begin{pmatrix} C_1^H \\ C_2^H \\ \cdot \\ \cdot \\ \cdot \\ C_n^H \end{pmatrix}$$

will be used frequently in the sequel. The condition

$$Im(T_{n-1}\ B_n) = Im(D_n\ T_{n-1}) \tag{I_n}$$

or equivalently

$$Ker \begin{pmatrix} T_{n-1} \\ B_n^H \end{pmatrix} = Ker \begin{pmatrix} D_n^H \\ T_{n-1} \end{pmatrix} \tag{18}$$

will play an important role. Note that for $n = 0$ T_{-1}, B_0 and D_0 are empty matrices, and (I_0) is trivially true.

After these preliminaries, we can state our main results.

Theorem 1. *The following conditions are equivalent:*

(a) T_n *admits a singular extension* T_{n+1}, *i.e.* $C(T_n) \neq \emptyset$.

(b) (I_n) *is satisfied.*

(c) The shift matrix U_0 *in (12) can be extended to a* T_n*-unitary matrix* $U = (U_0\ U_1)$, i.e. $U(T_n) \neq \emptyset$.

(d) $F(T_n) \neq \emptyset$, *i.e.* T_n *admits a factorization (14).*

Corollary 1. T_n *admits singular extensions and factorizations of the form (14), if one of the following conditions is satisfied:*

a) T_n *is positive semi-definite.*

b) T_{n-1} *is nonsingular.*

c) T_n *is nonsingular.*

d) $T_n(C_0, \ldots, C_n)$ *is an Hermitian block circulant matrix, i.e.* $C_0 = C_0^H$ *and* $C_j = C_{n+1-j}^H$, $j = 1, \ldots, n$.

The sufficiency of condition a) is a consequence of Remark 2. The condition b) as well as c) imply that $(T_{n-1}\ B_n)$ and $(D_n\ T_{n-1})$ are of full rank, and thus (I_n) holds. Finally, in case d) $B_n = D_n$ and (I_n) is trivially true.

Theorem 1 states that C, U and F are either all empty or all non-empty. There is even a one-to-one correspondence between C, U and F. For any T_n-unitary matrix $U = (U_0\ U_1)$, where U_0 is the shift matrix (12), we define

$$C_{n+1} = \tau(U) := (\underbrace{I}_{q}\ \underbrace{0}_{nq})\ T_n\ U_1. \tag{19}$$

For any factorization (14) of T_n, we set

$$C_{n+1} = \sigma(X, W) = X^H \Lambda W^{n+1} X. \tag{20}$$

Theorem 2.

a) Let (I_n) *be satisfied. Then, (19) and (20) define bijective mappings*

$$\tau : U(T_n) \to C(T_n)$$

and

$$\sigma : F(T_n) \to C(T_n).$$

b) (r_n) implies (I_n).
c) (r_n) is true iff $C(T_n)$ (and thus $U(T_n)$ and $F\ (T_n)$) contains precisely one element.

From Theorem 1 and parts b), c) of Theorem 2, we immediately obtain the following

Corollary 2. *Let (I_n) be satisfied. Then, T_n can be extended to an infinite block Toeplitz matrix $T_\infty\ (C_0, C_1, \ldots)$ such that*

$$\text{rank } T_{n-j} = \text{rank } T_n,\ 0 \leq j \leq \infty.$$

Moreover, the matrices C_{n+j}, $j \geq 2$, are uniquely determined by T_n and C_{n+1}. C_{n+1} is *uniquely determined by T_n iff (r_n) holds.*

Remarks 3. By means of the mappings τ and σ, the canonical representation (11) of $U\ (T_n)$ yields a description of all singular extensions and all factorizations (14) of T_n.

4. Several authors have considered the singular extension problem for special classes of matrices (1) using different techniques. Delsarte, Genin and Kamp [11] treated the problem for scalar ($q = 1$) Hermitian Toeplitz matrices. Fritzsche and Kirstein [14] solved the positive semi-definite block case completely. Constantinescu [7] studied extensions of $T_0 = C_0$.

5. For singular scalar Toeplitz matrices (1), it follows from [13, Thm. 4] that (I_n) and (r_n) are equivalent. In the block case, this is no longer true as the following example for $q = 2$ and $n = 1$ shows:

$$T_1 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \ \text{rank } T_1 = 3 \neq \text{rank } T_0 = 2.$$

4. Proof of the main results

Using the partitions (17), one immediately obtains the following

Lemma 1. *If T_n and T_{n-1} are of the same rank, then*

$$Im B_n \subset Im T_{n-1} \text{ and } Im D_n \subset Im T_{n-1}.$$

We now turn to the proof of Theorem 1. Let T_{n+1} be a singular extension of T_n. Lemma 1 (with n replaced by $n+1$) gives

$$Im \begin{pmatrix} C_{n+1} \\ B_n \end{pmatrix} \subset Im T_n \text{ and } Im \begin{pmatrix} D_n \\ C_{n+1}^H \end{pmatrix} \subset Im T_n,$$

and using (17), (I_n) follows. Thus (a) implies (b).

Now suppose that (I_n) holds. To prove (c), we apply Theorem B with $H = T_n$, $A = T_{n-1}$, U_0 as in (12), and therefore have to verify (10). It follows from (8) and (9) that

$$S^H T_n U_0 S_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{H} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad S^{-1} U_0 S_{11} = \begin{pmatrix} 0 & 0 \\ \tilde{H}\tilde{V}_0 & \tilde{H}\tilde{U}_0 \\ 0 & 0 \end{pmatrix}. \tag{21}$$

We partition

$$S_{11} = (\underbrace{R_1}_{d_0} \quad \underbrace{R_2}_{m_0+k})$$

accordingly. Then (7) and (8) yield $T_n \begin{pmatrix} R_1 \\ 0 \end{pmatrix} = 0$, and together with (18) one abtains

$$0 = \begin{pmatrix} T_{n-1} \\ B_n^H \end{pmatrix} R_1 = \begin{pmatrix} D_n^H \\ T_{n-1} \end{pmatrix} R_1 = T_n U_0 R_1.$$

Thus $\tilde{H}\tilde{V}_0 = 0$ in (21), and since $\tilde{H}$ is nonsingular, this proves the first part of (10). On the other hand, (7) and (8) show that

$$S^{-H} \begin{pmatrix} 0 & 0 \\ \Lambda_1 & 0 \\ 0 & 0 \\ 0 & I_k \\ 0 & 0 \end{pmatrix} = T_n \begin{pmatrix} R_2 \\ 0 \end{pmatrix} = \begin{pmatrix} T_{n-1} \\ B_n^H \end{pmatrix} R_2 =: G,$$

and therefore $rank\ G = m_0 + k$. In view of (I_n), this is equivalent to

$$rank \begin{pmatrix} D_n^H \\ T_{n-1} \end{pmatrix} R_2 = m_0 + k,$$

and from (21) we deduce $rank\ \tilde{U}_0 = m_0 + k$. This concludes the proof of "$(b) \Rightarrow (c)$".

Next we show that (c) implies (d). Assume that $U = (U_0\ U_1)$ is a T_n-unitary matrix. With

$$X^{(j)} := \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix} \begin{matrix} \}jq \\ \}q \\ \}(n-j)q \end{matrix} \qquad , j = 0, 1, \ldots, n,$$

one has $U_0 = (X^{(1)}\ X^{(2)} \ldots X^{(n)})$. Thus $U X^{(j)} = X^{(j+1)}, j = 0, \ldots, n - 1$, and

$$I = K_n(X^{(0)}, U). \tag{22}$$

From the eigenvalue decomposition of T_n, we obtain a nonsingular matrix V and an $r \times r$ matrix of the form (13), where $r = rank\ T_n$, such that

$$T_n = V^H \tilde{\Lambda} V, \ \tilde{\Lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}.$$

Together with (22) this yields the factorization

$$T_n = K_n(\tilde{X}^{(0)}, \tilde{W})^H \, \tilde{\Lambda} \, K_n(\tilde{X}^{(0)}, \tilde{W}) \tag{23}$$

where

$$V X^{(0)} =: \tilde{X}^{(0)} = \begin{pmatrix} X \\ \star \end{pmatrix} \}r \text{ and } VUV^{-1} =: \tilde{W} = \begin{pmatrix} W & Y \\ \star & \star \end{pmatrix} \}r.$$

$$\underbrace{}_{r}$$

The T_n-unitary of U leads to the identity

$$\tilde{W}^H \tilde{\Lambda} \tilde{W} = \tilde{\Lambda}$$

which implies $W^H \Lambda W = \Lambda$ and $Y = 0$. Then

$$\tilde{W}^k = \begin{pmatrix} W^k & 0 \\ \star & \star \end{pmatrix}, \ k = 0, 1, \ldots,$$

and (23) reduces to (14). Thus (d) holds.

Part a) of the following lemma shows that (d) implies (a), and this concludes the proof of Theorem 1.

Lemma 2.

a) $C_{n+1} = \sigma(X, W)$ *in (20) defines a singular extension* T_{n+1} *of* T_n.

b) Let $T_n = K_n(X_j, W_j)^H \Lambda K_n(X_j, W_j)$, $j = 1,2$, *be two given factorizations (14). Then, (16) and*

$$\sigma(X_1, W_1) = \sigma(X_2, W_2) \tag{24}$$

are equivalent.

Proof. a) From (14) and (20) we get

$$T_{n+1}(C_0, \ldots, C_{n+1}) = K_{n+1}(X, W)^H \Lambda K_{n+1}(X, W),$$

and, obviously, T_{n+1} is a singular extension of T_n.

b) (16) immediately gives (24). Now suppose that (24) holds, i. e.

$$\sigma(X_j, W_j) = X_j^H \Lambda W_j^{n+1} X_j =: C_{n+1}, j = 1, 2.$$

Then, in addition to the factorizations

$$T_n(C_0, \ldots, C_n) = K_n^{(1)^H} \Lambda K_n^{(1)} = K_n^{(2)^H} \Lambda K_n^{(2)}, \tag{25}$$

we have

$$T_{n+1}(C_0, \ldots, C_n, C_{n+1}) = K_{n+1}^{(1)^H} \Lambda K_{n+1}^{(1)} = K_{n+1}^{(2)^H} \Lambda K_{n+1}^{(2)}. \tag{26}$$

Here

$$K_m^{(j)} := K_m(X_j, W_j), j = 1, 2, m = n, n + 1.$$

Since K_n has full rank r, one can choose a nonsignular $r \times r$ submatrix R_1 of $K_n^{(1)}$, and let R_2 be the corresponding submatrix of $K_n^{(2)}$. (25) implies

$$R_1^H \Lambda R_1 = R_2^H \Lambda R_2;$$

therefore R_2 ist nonsingular and $Z := R_2 R_1^{-1}$ is Λ-unitary. R_j is also a submatrix of $K_{n+1}^{(j)}$ for $j = 1, 2$, respectively, and by using (26) we get

$$0 = R_1^H \Lambda K_{n+1}^{(1)} - R_2^H \Lambda K_{n+1}^{(2)} = R_1^H \Lambda (K_{n+1}^{(1)} - Z^{-1} K_{n+1}^{(2)}).$$

It follows that $K_{n+1}^{(2)} = Z K_{n+1}^{(1)}$, and since

$$K_{n+1}^{(j)} = (X_j \quad W_j K_n^{(j)}), j = 1, 2,$$

one obtains

$$X_2 = Z X_1 \text{ and } W_2 K_n^{(2)} = Z W_1 K_n^{(1)}.$$

By considerung the submatrices R_j of $K_n^{(j)}$, the second identity yields

$$W_2 = (W_2R_2)\,R_2^{-1} = (ZW_1R_1)\;R_2^{-1} = ZW_1Z^{-1}.$$

Thus both conditions in (16) are satisfied.

In the rest of this section, we prove Theorem 2. Part b) is an immediate consequence of Lemma 1. Now we turn to part a) and assume that (I_n) holds. Note that Theorem 1 guarantees that U, C and F are not empty. First we show that $\sigma: F(T_n) \to C(T_n)$ is bijective. By part a) of Lemma 2 $\sigma(X, W) \in C\;(T_n)$ for each factorization (14); moreover, by part b) two (non-)equivalent factorizations lead (not) to the same singular extension. Thus (20) defines an injective mapping on $F(T_n)$. Let $C_{n+1} \in C(T_n)$ be given. Then $T_n(C_0, \ldots, C_n)$ and $T_{n+1}(C_0, \ldots, C_{n+1})$ satisfy (r_{n+1}) and hence (I_{n+1}) by part b) of Theorem 2. In view of Theorem 1, there exists a factorization of type (14) for T_{n+1}:

$$T_{n+1} = K_{n+1}(X, W)^H \Lambda\, K_{n+1}\,(X, W).$$

In particular,

$$T_n = K_n(X, W)^H \Lambda\, K_n(X, W) \text{ and } C_{n+1} = \sigma(X, W).$$

This shows the surjectivity of σ, and in all the mapping σ is one-to-one.

Next we consider τ. From (12) and the first partition in (17), it follows that $U = (U_0\; U_1)$ is T_n-unitary iff

$$B_n = (\underbrace{\;0\quad I\;}_{q\quad nq})\; T_nU_1 \text{ and } C_0 = U_1^H\, T_nU_1. \tag{27}$$

Now let $U = (U_0\; U_1)$ and $U' = (U_0\; U_1')$ be two T_n-unitary matrices with canonical representations (11). From (7), (8) and (11), we get

$$T_n(U_1 - U_1') = S^{-\mathrm{H}} \begin{pmatrix} 0 & 0 \\ \tilde{H}L(M-M') & 0 \\ 0 & 0 \end{pmatrix} S_{22}^{-1}$$

where $\tilde{H}L$ is nonsingular. Thus U and U' are equivalent iff

$$T_nU_1 = T_n\, U_1'. \tag{28}$$

However, by the first relation in (27), the last nq equations in (28) are always fulfilled. Thus, by (19), (28) is equivalent to $\tau(U) = \tau(U')$.

This shows that (19) defines an injective mapping on $U(T_n)$. To prove $C_{n+1} = \tau(U) \in C(T_n)$, we note that by (19) and (27),

$$T_{n+1}(C_0, \ldots, C_{n+1}) = \begin{pmatrix} T_n & B_{n+1} \\ B_{n+1}^H & C_0 \end{pmatrix} = \begin{pmatrix} T_n & T_n U_1 \\ U_1^H T_n & U_1^H T_n U_1 \end{pmatrix}.$$

Hence

$$T_{n+1}(C_0, \ldots, C_{n+1}) = \begin{pmatrix} I & 0 \\ U_1^H & I \end{pmatrix} \begin{pmatrix} T_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & U_1 \\ 0 & I \end{pmatrix} \tag{29}$$

is a singular extension of T_n. Conversely, if $C_{n+1} \in C(T_n)$ is given, Lemma 1 (with n replaced by $n + 1$) ensures the existence of a matrix U_1 such that

$$B_{n+1} = \begin{pmatrix} C_{n+1} \\ B_n \end{pmatrix} = T_n U_1,$$

i.e. $U = (U_0 \ U_1)$ statisfies (19) and the first identity of (27). Moreover, from *rank* T_{n+1} = *rank* T_n we deduce that (29) holds. This implies $C_0 = U_1^H T_n U_1$, and, in view of (27), U is T_n-unitary. Thus τ is surjective, and the proof of a) is complete.

It remains to prove part c) of Theorem 2. We show that (r_n) is equivalent to $|U(T_n)| = 1$. Note that, by part b) of Theorem 2 and Theorem 1, (r_n) guarantees $U(T_n) \neq \emptyset$. From the canonical representation (11) we see that $U(T_n)$ consists of precisely one equivalence class iff $k = l = 0$ in (8) (with $H = T_n$, $A = T_{n-1}$). This is equivalent to the rank condition (r_n).

5. Generalizations of Carathéodory's theorem

Let $T_n(C_0, \ldots, C_n)$ be an Hermitian block Toeplitz matrix (1) of rank r and Λ be the signature matrix (13) where μ and ν is the number of positive and negative eigenvalues of T_n, respectively. Each factorization of the form (14) with X $r \times q$ and W an $r \times r$ Λ-unitary matrix is equivalent to the representation

$$C_k = X^H \Lambda W^k X, \ k = 0, \ldots, n, \tag{30}$$

of the blocks of T_n. By transforming Λ and W to a certain normal form, we now deduce from (30) generalizations of Carathéodory's Theorem A.

First, some needed results from the theory of H-unitary matrices (e. g. Gohberg, Lancaster and Rodman [15]) are briefly recalled. Let H be a nonsingular Hermitian matrix and U H-unitary. The spectrum of U is symmetric relative to the unit circle; i. e. if λ is an eigenvalue, so is $\bar{\lambda}^{-1}$, and Jordan blocks corresponding to symmetric pairs are of the same size [15, p. 26]. Thus U has Jordan normal form

$$J = diag(J_{d_1}(\lambda_1), \ldots, J_{d_\alpha}(\lambda_\alpha), J_{d_{\alpha+1}}(\lambda_{\alpha+1}), J_{d_{\alpha+1}}(\bar{\lambda}_{\alpha+1}^{-1}), \ldots, J_{d_{\alpha+\beta}}(\lambda_{\alpha+\beta}), J_{d_{\alpha+\beta}}(\bar{\lambda}_{\alpha+\beta}^{-1})) \tag{31}$$

where

$$J_l(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \ldots & \ldots & 0 & \lambda \end{pmatrix}$$

denotes the $l \times l$ Jordan block and

$$\lambda_j = \begin{cases} e^{i\Phi_j}, j = 1, \ldots, \alpha \\ s_j e^{i\Phi_j}, 0 < s_j < 1, j = \alpha + 1, \ldots, \alpha + \beta, \end{cases} \tag{32}$$

and we always assume $0 \leq \Phi < 2\pi$. From (31) results a corresponding normal form of the pair(U, H) composed of upper triangular Toeplitz matrices of the types

$$N_l = \begin{pmatrix} 1 & 2i & 2i^2 & \ldots & 2i^{l-1} \\ 0 & 1 & 2i & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & 2i^2 \\ \vdots & & \ddots & \ddots & 2i \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} \tag{33a}$$

$$N_l(\lambda) = \begin{pmatrix} 1 & \varkappa_1 & \varkappa_2 & \ldots & \varkappa_{l-1} \\ 0 & 1 & \varkappa_1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \varkappa_2 \\ \vdots & & \ddots & \ddots & \varkappa_1 \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}, \; N_l'(\lambda) = \begin{pmatrix} 1 & \chi_1 & \chi_2 & \ldots & \chi_{l-1} \\ 0 & 1 & \chi_1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \chi_2 \\ \vdots & & \ddots & \ddots & \chi_1 \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix}, \tag{33b}$$

where for $\lambda \neq 0$, $j = 1, 2, \ldots, 1-1$

$$\varkappa_j = q_1^{j-1}(q_1 - \bar{q}_2), \quad \chi_j = q_1^{j-1}(q_2 - \bar{q}_1)$$

with

$$q_1 = \frac{i}{2}\,(1 + \lambda), \quad q_2 = \frac{i}{2}\,(1 + \bar{\lambda}^{-1}),$$

and of $l \times l$ anti-identity matrices

$$P_l = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \tag{34}$$

Note that N_l is P_l-unitary and *diag* $(N_l(\lambda),\ N_l'(\lambda))$ is unitary with respect to P_{2l}; moreover, $N_l'(\lambda) = \bar{N}_l^{-1}$ where $\bar{N}_l$ denotes the matrix whose entries are the complex conjugates of those of N_l [15, p. 26].

Theorem C ([**15, Thm.** *4.1*]). *Let H be a nonsingular Hermitian matrix and U be H-unitary with Jordan normal form J arranged as in (31), (32). Then, there exists a nonsingular matrix S and an ordered set $\epsilon = \{\epsilon_1, \dots, \epsilon_\alpha\}$ of signs $\pm$ 1 such that*

$$U = S^{-1} N_J\, S \text{ and } H = S^H P_{\epsilon, J}\, S$$

where

$$\mathrm{N_J} = \mathit{diag}(N_{d_1}, \dots, N_{d_\alpha}, N_{d_{\alpha+1}}(\lambda_{\alpha+1}), N'_{d_{\alpha+1}}(\bar{\lambda}_{\alpha+1}^{-1}), \dots, N_{d_{\alpha+\beta}}(\lambda_{\alpha+\beta}), N'_{d_{\alpha+\beta}}(\bar{\lambda}_{\alpha+\beta}^{-1})) \tag{35}$$

and

$$\mathrm{P}_{\epsilon, J} = \mathit{diag}\,(\epsilon_1 P_{d_1}, \dots, \epsilon_\alpha P_{d_\alpha}, P_{2d_{\alpha+1}}, \dots, P_{2d_{\alpha+\beta}}). \tag{36}$$

Remark that N_J is $P_{\epsilon,J}$-unitary.

We now apply Theorem C to the Λ-unitary matrix W, and thus from (30) one obtains

$$C_k = Y^H P_{\epsilon,J}\, N_J^k\, Y,\ k = 0, \dots, n, \tag{37}$$

where $Y := S\,X$. Partitioning

$$Y = (Y_1^H, Y_2^H, \dots, Y_\alpha^H, Y_{\alpha+1}^H, (Y'_{\alpha+1})^H, \dots, Y_{\alpha+\beta}^H, (Y'_{\alpha+\beta})^H)^H \tag{38}$$

in conformity with (35) and rewriting of (37) in terms of the blocks of (35) and (36), then leads to the following Carathéodory type representation: For $k = 0, \dots n$

$$C_k = \sum_{j=1}^{\alpha} \epsilon_j e^{i\Phi_j k} \varrho_j^{(k)} + \sum_{j=\alpha+1}^{\alpha+\beta} e^{i\Phi_j k}(s_j^k \Psi_j^{(k)}(s_j e^{i\Phi_j}) + s_j^{-k}\Omega_j^{(k)}(s_j e^{i\Phi_j})). \qquad (39)$$

Here the abbreviations

$$\varrho_j^{(k)} = Y_j^H P_{d_j} N_{d_j}^k Y_j, \quad \Psi_j^{(k)}(\lambda) = (Y_j')^H P_{d_j} N_{d_j}(\lambda)^k Y_j, \qquad (40a)$$

$$\Omega_j^{(k)}(\lambda) = Y_j^H P_{d_j} N'_{d_j}(\lambda)^k Y_j' \qquad (40b)$$

are used. Note that

$$(d :=) \sum_{j=1}^{\alpha+\beta} d_j = rank\ T_n. \qquad (41)$$

Conversely, if a representation (39) is given, by defining N_J, $P_{\epsilon,J}$ and Y via (35), (36) and (38), we arrive at the factorization

$$T_n(C_0, \ldots C_n) = K_n(Y, N_J)^H P_{\epsilon,J} K_n(Y, N_J). \quad (42)$$

This shows that a representation (39) with $d < rank\ T_n$ is not possible. Moreover, if (41) holds, then (42) can easily be transformed into a factorization (14). Thus, in view of Theorem 1, we have proved the following

Theorem 3. *Let $T_n(C_0, \ldots, C_n)$ be an Hermitian block Toeplitz matrix. Then, there are equivalent:*
(a) (I_n) is satisfied.
(b) There exists a representation (39), (40) of the blocks $C_0, \ldots, C_n$ with $0 \le \Phi_j < 2\pi$, $j = 1, \ldots, \alpha + \beta$, $0 < s_j < 1$, $j = \alpha + 1, \ldots, \alpha + \beta$, and components of type (33), (34), and the additional requirement (41) is satisfied.

The representation (39) simplifies considerably if the associated Jordan normal form (31) is a diagonal matrix, and one obtains the following

Theorem 4. *Let $T_n(C_0, \ldots, C_n)$ be an Hermitian block Toeplitz matrix with $r := rank\ T_n > 0$.*

a) If T_n has a factorization (14) with an $r \times r$ diagonalizable Λ-unitary matrix W, then the blocks of T_n can be represented in the form

$$C_k = \sum_{j=1}^{t_1} e^{i\Phi_j k}\varrho_j - \sum_{j=t_1+1}^{t_2} e^{i\Phi_j k}\varrho_j + \sum_{j=t_2+1}^{t_3} {}^{i\Phi_j k}(s_j^k \Psi_j + s_j^{-k} \Psi_j^H),$$

$$k = 0, \ldots, n, \qquad (43)$$

where

$$0 \leq \Phi_1 < \Phi_2 < \ldots < \Phi_{t_1} < 2\pi,$$

$$0 \leq \Phi_{t_1+1} < \Phi_{t_1+2} < \ldots < \Phi_{t_2} < 2\pi,$$

$0 \leq \Phi_j < 2\pi$, $0 < s_j < 1$ *and all* $s_j e^{i\Phi_j}$ *are distinct,* $j = t_2 + 1, \ldots, t_3$,

and the matrices ϱ_j, $j = 1, \ldots, t_2$, *are Hermitian and positive semi-definite, and*

$$\mu := \sum_{j=1}^{t_1} \text{rank } \varrho_j + \sum_{j=t_2+1}^{t_3} \text{rank } \Psi_j \tag{44a}$$

and

$$\nu := \sum_{j=t_1+1}^{t_2} \text{rank } \varrho_j + \sum_{j=t_2+1}^{t_3} \text{rank } \Psi_j \tag{44b}$$

are just the numbers of positive and negative eigenvalues of T_n, *respectively. Moreover, there are no representations (43) with* $\mu + \nu < r$.

b) If T_n *is positive semi-definite, there exists an integer* t, *numbers* $0 \leq \Phi_1 < \Phi_2 < \ldots < \Phi_t < 2\pi$, *and positive semi-definite Hermitian matrices* ϱ_j, $j = 1, \ldots t$, *such that*

$$C_k = \sum_{j=1}^{t} e^{i\Phi_j k} \varrho_j, \quad k = 1, \ldots, n, \tag{45}$$

and

$$r = \sum_{j=1}^{t} \text{rank } \varrho_j.$$

The representation (45) is unique iff the rank condition (r_n) *is satisfied.*

Proof. a) Let (39) be the representation induced by the diagonalizable Λ-unitary matrix W. Then, $d_j = 1, j = 1, \ldots, \alpha + \beta$, and since $P_1 = N_1 = N_1(\lambda) = N_1'(\lambda) = (1)$ the matrices (40) are independent of k and λ. Moreover, $\varrho_j^{(k)} = Y_j^H Y_j$ is positive semi-definite, $j = 1, \ldots, \alpha$, and $\Omega_j^{(k)} = (\Psi_j^{(k)})^H$, $j = \alpha + 1, \ldots, \alpha + \beta$. By collecting terms in (39) with coinciding ϵ_j and Φ_j, $j = 1, \ldots, \alpha$, and coinciding numbers $s_j e^{i\Phi_j}$, $j = \alpha + 1, \ldots, \alpha + \beta$, respectively, and by a possible renumbering, we obtain a representation (43). Since Λ and $P_{\epsilon,J}$ have the same signature, one gets $\mu \leq \mu_+$ and $\nu \leq \nu_-$. Here, μ and ν are defined by (44), and μ_+ (ν_-) denotes the number of positive (negative) eigenvalues of T_n. It remains to show that $\mu + \nu < r$ is impossible.

Assume that (43) is given. There exist matrices Y_j and Y_j' of full column rank such that

$$\varrho_j = Y_j^H Y_j, j = 1, \ldots, t_2, \quad \Psi_j = (Y_j')^H Y_j, \quad j = t_2 + 1, \ldots, t_3,$$

and define Y via (38) ($\alpha = t_2$, $\beta = t_3 - t_2$). Note that Y has $\mu + \nu$ rows. It is easily verified that (43) can be rewritten as a factorization of type

$$T_n = K_n(Y, D)^H P K_n(Y, D), \tag{46}$$

and this implies $\mu + \nu \geq r$.

b) Let T_n be positive semi-definite. By Corollary 1, T_n admits a factorization (14) with $\Lambda = I$ and W a unitary matrix. Hence, W is diagonalizable, and there exits a representation (43) which, in view of (44), reduces to (45). It remains to show that the representation (45) is unique iff (r_n) holds. To this end, recall that (45) is equivalent to the factorization (46) with $P = I$ and

$$D = diag(e^{i\Phi_1} I_{l_1}, \ldots, e^{i\Phi_t} I_{l_t})$$

where $l_j = rank\ \varrho_j = rank\ Y_j$, $j = 1, \ldots, t$. As a simple calculation shows, the representation (45) is unique iff these factorizations are equivalent in the sense of (16). By part c) of Theorem 2 this is equivalent to (r_n).

Remarks 6: For the special case of scalar Toeplitz matrices, the representation (43) was already derived by Delsarte and Genin [10] using a technique different from our approach.

7. (44) can be rewritten in the form

$$C_k = \int_0^{2\pi} (e^{i\Phi})^k \, d\varrho(\Phi), \quad k = 0, \ldots, n,$$

with ϱ defined by

$$\varrho(\Phi) := \sum_{j=1}^{l} \varrho_j \text{ for } \Phi_l \leq \Phi < \Phi_{l+1}, \quad l = 0, \ldots, t,$$

($\Phi_0 := 0$, $\Phi_{t+1} := 2\pi$). The matrix-valued function $\varrho(\Phi)$ is Hermitian and nondecreasing in the sense that $\varrho(\Phi') - \varrho(\Phi)$ is positive semi-definite for $\Phi' \geq \Phi$. Thus, as a by-product, we have obtained an elementary proof of the solvability of the truncated trigonometric moment problem for matrix-valued measures. This problem (for the more general operator version) was first solved by Ando [3] using the Naimark Dilation Theorem.

8. In general, (I_n) does not imply that T_n admits a representation (43). Consider the scalar Toeplitz matrix

$$T_2(c_0, c_1, c_2) \text{ with } c_k = ki, \quad k = 0, 1, 2, \quad i^2 = -1.$$

It is easily verified that a representation (43) is not possible. However (I_2) holds, and the entries of T_2 can be written in the form (39): For $k = 0, 1, 2$

$$c_k = Y_2^H P_2 N_2^k Y_2, \ Y := \begin{pmatrix} i \\ 1/\sqrt{2} \end{pmatrix}.$$

Finally, we state an extension of Carathéodory's Theorem A for the representation of arbitrary complex matrices.

Corollary 3. *Let $C_1, \ldots, C_n$ be given complex $q \times q$ matrices not all zero, $n \geq 1$. Then, to each Hermitian positive definite $q \times q$ matrix Σ, there exists a minimal integer r, an integer $t = t(r)$, numbers Φ_j with*

$$0 \leq \Phi_1 < \Phi_2 < \ldots < \Phi_t < 2\pi,$$

Hermitian positive semi-definite matrices ϱ_j, $j = 1, \ldots, t$, and a number $\sigma > 0$ such that

$$C_k = \sum_{j=1}^{t} e^{i\Phi_j k} \varrho_j, \quad k = 1, \ldots n, \tag{47}$$

and

$$\sigma\Sigma = \sum_{j=1}^{t} \varrho_j, \quad r = \sum_{j=1}^{t} \operatorname{rank} \varrho_j. \tag{48}$$

Moreover, $r = \operatorname{rank} T_n(\sigma\Sigma, C_1, \ldots, C_n)$ and the representation (47), (48) is unique iff T_n satisfies the rank condition (r_n).

This corollary follows immediately from part b) of Theorem 4 applied to $T_n(C_0, \ldots, C_n)$. Here $C_0 = \sigma\Sigma$ where $\sigma > 0$ is the uniquely determined number such that $T_n(\sigma\Sigma, C_1, \ldots, C_n)$ is positive semi-definite, but singular.

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