BAYERISCHE AKADEMIE DER WISSENSCHAFTEN
MATHEMATISCH-NATURWISSENSCHAFTLICHE KLASSE

SITZUNGSBERICHTE

JAHRGANG

1981

MÜNCHEN 1982

VERLAG DER BAYERISCHEN AKADEMIE DER WISSENSCHAFTEN
In Kommission bei der C. H. Beck'schen Verlagsbuchhandlung München

Duality of Vector-valued (al)-Norms and (am)-Norms on Ordered Vector Spaces.

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in der Sitzung vom 3. Juli 1981 für SB II

§ 1. **Introduction.** The dual norm of a scalar absolute (L)-norm, i. e. an (AL)-norm on a vector lattice, is an absolute (M)-norm or an (AM)-norm, and more precisely an order-unit norm. Conversely, an absolute (M)-norm has as its dual an absolute (L)-norm. These and other well-known results about (AL)-norms and (AM)-norms go back to Kakutani [9] and have been generalized in various ways. Two important directions, in which generalizations have been made, were shown by Edwards [5] and Ellis [6] to the one side and F. L. Bauer [1] to the other.

Edwards and Ellis introduced the notion of a "cone-base" norm, proved theorems concerning the duality of order-unit and cone-base norms and there by generalized results of Kakutani about norms on vector lattices in the sense of more general orderings on the space the norms are defined on. On the other hand Bauer considered vector-valued norms on vector lattices. He formally carried over the notions of an (AL)-norm and an (AM)-norm from the scalar case to the case of vector-valued norms, introduced an appropriate notion of regularity for vector-valued norms and proved [1] among other things that a regular (AL)-norm possesses as its dual a vector-valued "order-interval" norm, i. e. the corresponding generalization of order-unit norms to the case of vector-valued norms. The introduction of the notion of regularity is due to the fact that not every vector-valued norm has a reasonable dual norm. A scalar norm always possesses a dual norm and therefore is regular. Accordingly, F. L. Bauer obtains one of the theorems of Kakutani mentioned above as a special case of his theorem about vector-valued norms. Bauer also raised questions, which do not appear in the scalar case. For instance, he asked for a necessary and sufficient condition for the regularity of (AL)-norms and proved [1] a "pretty good" necessary condition for the regularity of an (AL)-norm. We will show in theorem 8.7 that this condition is also sufficient.

The aim of the present paper is to bring together the two directions mentioned above, in which the results of Kakutani have been generalized, i. e. to define comprehensive notions and to prove corresponding duality theorems. This task is accomplished with the concept of vector-valued absolute-monotone (l)-norms and (m)-norms, (al)-norms and (am)-norms in short, on ordered vector spaces in the following sense. The (al)- and (am)-norms are identical with the classical (AL)- and (AM)-norms, respectively, if the underlying ordered vector space is a vector lattice (§ 8). In the case of scalar norms on ordered vector spaces we have the following relations. Every (al)-norm is a cone-base norm and every cone-base norm which is itself the dual of an "approximate order-unit" norm [12] is also an (al)-norm. Similarly a scalar (am)-norm is always an approximate order-unit norm and every order-unit norm is an (am)-norm. The (al)- and (am)-norms can be generalized without difficulty in such a way that in the scalar case the (al)-norms are even identical with the cone-base norms and the (am)-norms with the approximate order-unit norms. To this end one has to substitute in some definitions the relation '$\leq$' with the relation '$<$', i. e. $\leq$ and $\neq$, or to require that some properties usually required for the closed norm balls should hold only for "weakly-open" norm balls [11]. Corresponding generalizations of (al)- and (am)-norms will be treated in a subsequent paper of the author entitled 'Duality of vector-valued monotonic norms on ordered vector spaces', in which duality theorems beyond the above scope are proved by topological means.

Let us summarize the contents of the paper succinctly. From §§ 3–5 we develop the new concepts and duality theorems. In § 8 to the discussion about vector-valued norms on vector lattices there will be added some new results. Before coming to the duality theorems we prove new theorems about general regular norms and introduce the new notion of a 'complete regular' norm

(§ 2). The central notion of an order-interval norm is investigated in § 6. There the close relation between order-interval norms, generalized absolute value operators and also sublattices of ordered vector spaces will become apparent. Conditions for regularity of (al)- and (am)-norms are the object of § 7.

Notation. Throughout we use the following notations. V^* is the space of all linear functionals on a vector space V. If $M \subset V$, Lin(M) denotes the subspace of V generated by M. Let (V, ϱ) be an ordered vector space. In context with an order relation we use $\leq$ in the usual meaning. If $x, y, z \in V$, we frequently write $x, y \leq z$ instead of $x \leq z$ and $y \leq z$. C^0_ϱ denotes the cone of all semipositive functionals contained in V^*. $V^0 := C^0_\varrho - C^0_\varrho$. $f \leq g$ for $f, g \in V^*$ means $f(x) \leq g(x), \forall x \in C_\varrho$, where $C_\varrho := \{x \in V : 0 \leq x\}$. The ordering determined by C^0_ϱ is called the dual ordering of ϱ in V^*, V^0 the order dual space of (V, ϱ). The composition of mappings f and g is denoted by $f \cdot g$. The natural ordering of functionals on a set M is denoted by $\leq_*$. We have $f \leq_* g$, if $f(x) \leq g(x), \forall x \in M$. $\inf_* A$ and $\inf_0 A$ denote the infimum of the elements of a set A with respect to the natural and dual ordering, respectively.

This paper is dedicated to my dear teacher, Professor Dr. Dr. h.c. Friedrich L. Bauer, who gave the impulse for it.

§ 2. Regular and completely regular norms.

Let V be a vector space over the real scalar field R and (H, σ) an ordered vector space over R with C_σ as its positivity cone. A *norm* p on V with values in H is a positive-definite symmetric sublinear mapping of V into H, i. e. a mapping with the following properties.

(p–d) $\quad 0 \leq p(x) \quad$ and $\quad (p(x) = 0 \Rightarrow x = 0)$,

(s) $\quad p(x) = p(-x)$,

(sl) $\quad p(x + y) \leq p(x) + p(y), \;\; p(\lambda x) = \lambda p(x)$

for all $x, y \in V$, $\lambda \in$ R, $0 \leq \lambda$. Together with a norm p one often considers the indexed family K_p of its *norm balls* or [11] *inverse* *predomains* $K_p[\gamma] := \{x \in V : p(x) \leq \gamma\}$ for all $\gamma \in H$. $p(x) = \inf \{\gamma \in H : x \in K_p[\gamma]\}$ holds. It is clear that in general a norm is not determined by one single norm ball, as it is the case for scalar norms. If $\gamma \notin C_\sigma$, then $K_p[\gamma] = \emptyset$.

As is usually done for scalar norms, we call a norm p^{d} the 'dual' norm of p, if it associates to each bounded linear functional on V a linear 'least upper bound'. We shall see that in the general case of vector-valued norms exactly the regular norms have a dual norm.
However one gets to different notions of regularity and correspondingly to different dual norms, depending on which mappings of H or C_σ one admits as upper bounds for functionals from V^*. In the following we define 'regular' norms and 'completely regular' norms. In the first case all the semipositive linear functionals are allowed as upper bounds, in the second case all the semipositive superlinear mappings of C_σ in R.

We suppose throughout that C_σ is generating, i. e. $C_\sigma - C_\sigma = H$. Let p be an H-valued norm on V. We call $\gamma' \in C_\sigma^0$ a *positive linear upper bound* of $f \in V^*$ and correspondingly f *bounded* by γ', if $f(x) \leq \gamma' \cdot p(x)$ for all $x \in V$. The set of all positive linear upper bounds of f is denoted by $B_p[f]$, $B_p[f] := \{\gamma' \in C_\sigma^0 : f(x) \leq \gamma' \cdot p(x), \forall\, x \in V\}$. Exactly for all bounded $f \in V^*$ we have $B_p[f] \neq \emptyset$. The set of all bounded linear functionals from V^* forms a subspace V^{b} of V^*. p is called *regular*, if for all $f \in V^{\mathrm{b}}$ the set of bounds $B_p[f]$ contains a least element with respect to the dual ordering of σ in H^0. Since a least element of $B_p[f]$ is uniquely determined and is itself the infimum of $B_p[f]$, we have

(2.1) p is regular $\asymp$ there is a unique mapping $p^{\mathrm{d}} : V^{\mathrm{b}} \to C_\sigma^0$, such that for all $f \in V^{\mathrm{b}}$

(i) $p^{\mathrm{d}}(f) = \inf_0 B_p[f]$ and

(ii) $p^{\mathrm{d}}(f) \in B_p[f]$, i. e. $f(x) \leq p^{\mathrm{d}}(f) \cdot p(x), \forall\, x \in V$ (*Hölder inequality*) and $0 \leq p^{\mathrm{d}}(f)$.

That p^{d} is a norm is easily verified. p^{d} is called the *dual* norm of p. The following equivalence is sometimes useful. Let $\nu : V^{\mathrm{b}} \to C_\sigma^0$.

(2.2) p is regular and ν is the dual norm of $p \bowtie$
$\forall f \in V^b, \gamma' \in C_\sigma^0 : (\nu(f) \leq \gamma' \Leftrightarrow \gamma' \in B_p[f])$, i. e.
$(f \in K_\nu[\gamma'] \Leftrightarrow \gamma' \in B_p[f])$.

The next description of regularity does not contain p^d explicitly.

(2.3) p is regular $\bowtie \forall f \in V^b : \exists \gamma' \in H^0 : \gamma' = \inf_0 B_p[f]$ and $\gamma' \in B_p[f]$.

Each downwards directed set B of semipositive linear functionals on H possesses an infimum for which, for all $\gamma \in C_\sigma$, $(\inf_0 B)(\gamma) = \inf\{\gamma'(\gamma) : \gamma' \in B\}$ (c. f. the proof of theorem 2.4). More generally this means that the order dual space H^0 of H is 'directed-complete'. An ordered vector space is defined to be *directed-complete*, if each subset which is directed downwards and bounded from below possesses an infimum. Because of the above representation of these infima we get from property 2.3 a convenient necessary and sufficient condition for regularity.

(2.4) **Theorem.** p is regular $\bowtie \forall f \in V^b : B_p[f]$ is directed downwards

⌊$<$: For all $\gamma \in C_\sigma$ there exists $u_f(\gamma) := \inf\{\gamma'(\gamma) : \gamma' \in B_p[f]\}$. The additivity of the functional u_f on C_σ follows from $u_f(\gamma) + u_f(\beta) = \inf\{\gamma'(\gamma) + \beta'(\beta) : \gamma', \beta' \in B_p[f]\} \leq \inf\{\gamma'(\gamma + \beta) : \gamma' \in B_p[f]\} \leq \inf\{\gamma'(\gamma) + \beta'(\beta) : \gamma', \beta' \in B_p[f]\} = u_f(\gamma) + u_f(\beta)$. Here the directedness of $B_p[f]$ is only needed for the proof of the second inequality. u_f is also positive-homogeneous and possesses therefore a linear extension $\bar{u}_f$ on H. Since for all $\gamma' \in B_p[f]$ and $x \in V$ the inequality $f(x) \leq \gamma' \cdot p(x)$ holds, $f(x) \leq \inf\{\gamma' \cdot p(x) : \gamma' \in B_p[f]\} = u_f(p(x))$, i. e. $\bar{u}_f \in B_p[f]$. Apparently we also have $\bar{u}_f = \inf_0 B_p[f]$, i. e. p is regular.⌋

The notion of 'complete regularity' is narrower, but possesses many applications. Completely regular are the scalar norms, the absolute-value mapping on a vector lattice, the norms with the decomposition property [4], „spaltbare" norms [7] as well as regular norms in the sense of Robert [13] or Bode [3]. Interesting is also that the dual norm of a regular (AL)-norm on a vector lattice is always completely regular (c. f. theorem 8.8). Several somewhat differing notions and applications thereof will be discussed in a subsequent paper 'Regular vector-valued norms' by the author.

We consider a semipositive superlinear functional q on C_σ, i. e. a mapping $q : C_\sigma \to \mathrm{R}$ with the properties $0 \leq q(x)$ and $q(x) + q(y) \leq q(x+y)$, $q(\lambda x) = \lambda q(x)$ for all $x, y \in C_\sigma$, $\lambda \in \mathrm{R}$, $0 \leq \lambda$. q is called an *upper bound* of $f \in V^*$, if $f(x) \leq q(p(x))$, $\forall x \in V$. The set of all semipositive superlinear upper bounds of $f \in V^b$ is denoted by $S_p[f]$. Clearly the functionals from $B_p[f]$ restricted to C_σ are elements of $S_p[f]$. But in contrast to $B_p[f]$ $S_p[f]$ always contains a least element with respect to the natural ordering $\leq_*$ of the set of the functionals on C_σ, namely $u_f := \inf_{\leq_*} S_p[f]$ or $u_f(\gamma) = \inf \{q(\gamma) : q \in S_p[f]\}$, $\forall \gamma \in C_\sigma$. This will be shown in the following theorem. Let again K_p be the indexed family of the norm balls of p.

(2.5) **Theorem.** Let $f \in V^b$. $u_f := \inf_{\leq_*} S_p[f]$ is the least element of $S_p[f]$, i. e. $u_f \in S_p[f]$. u_f can be represented as follows.

$u_f(\gamma) = \inf \{q(\gamma) : q \in S_p[f]\} = \sup \{f(x) : x \in K_p[\gamma]\}$, $\forall \gamma \in C_\sigma$.

[$(\inf_{\leq_*} S_p[f])(\gamma) = \inf \{q(\gamma) : q \in S_p[f]\}$ is obtained from the definition of the natural ordering $\leq_*$. There remains the 2. equality to be shown. The set $\{f(x) : x \in K_p[\gamma]\}$ is bounded by $q(\gamma)$ for all $q \in S_p[f]$. Because of the definition we have for all $q \in S_p[f]$ $f(x) \leq q(p(x))$, $\forall x \in V$. But a semipositive superlinear functional q is also isotone. Therefore from $f(x) \leq q(p(x))$ for all γ with $p(x) \leq \gamma$, i. e. for all γ such that $x \in K_p[\gamma]$ follows also $f(x) \leq q(\gamma)$. We define $v_f(\gamma) := \sup \{f(x) : x \in K_p[\gamma]\}$ for all $\gamma \in C_\sigma$. Clearly $v_f \leq_* u_f$. From the sublinearity of p follows $\lambda K_p[\gamma] + \mu K_p[\beta] \subset K_p[\lambda\gamma + \mu\beta]$ for all $\gamma, \beta \in C_\sigma$, $\lambda, \mu \geq 0$, and therefrom the superlinearity of v_f. For all $\gamma \in C_\sigma$ $0 = p(0) \leq \gamma$, i. e. $0 \in K_p[\gamma]$. Therefore $0 \leq v_f(\gamma)$, $\forall \gamma \in C_\sigma$, i. e. v_f is semipositive. We obtain $v_f \in S_p[f]$ and

$u_f \leq_* v_f$. Now $u_f = v_f$ is proved. Clearly u_f is the least element of $S_p[f]$.]

Suppose p is regular. Then the semipositive superlinear mapping u_f describes the same set of semipositive linear upper bounds of a functional $f \in V^b$ as p^d. That is the content of the following theorem, which is easily verified.

(2.6) **Theorem.** Let p be regular. Then for all $f \in V^b$ and $\gamma' \in C_\sigma^0$ we have $p^d(f) \leq \gamma' \Leftrightarrow u_f(\gamma) \leq \gamma'(\gamma)$, $\forall\, \gamma \in C_\sigma$.

An H-valued norm on V is called *completely regular*, if for all $f \in V^b$ the least semipositive superlinear upper bound of f is linear. Let $B_p^r[f]$ denote the set of all functionals from $B_p[f]$ restricted to C_σ. Then the complete regularity of p is equivalent to the property $\forall\, f \in V^b : \inf_{\leq_*} S_p[f] \in B_p^r[f]$. Therefrom is easily obtained

(2.7) **Lemma.** If p is completely regular, then also p is regular and we have for all $f \in V^b$, $\gamma \in C_\sigma$

$p^d(f)\,(\gamma) = \sup\,\{f(x) : x \in K_p[\gamma]\}$, therefore
$\sup\,\{f(x) : x \in K_p[\gamma]\} = \inf\,\{\gamma'(\gamma) : \gamma' \in B_p[f]\}$.

§ 3. (l)-norms and (m)-norms.

In this paragraph and all the following ones we consider vector-valued norms on ordered vector spaces. We introduce the concepts of an (l)-norm and an (m)-norm and show, that the dual norm of a regular (l)-norm is an (m)-norm. Conversely, the dualization of certain (m)-norms leads to (l)-norms.
Let (V, ϱ) and (H, σ) be ordered vector spaces over R. The positivity cones C_ϱ in V and C_σ in H are always supposed to be generating. Let p be an H-valued norm on V. p is an *(l)-norm*, if p is *additive* over C_ϱ, i.e. if

(l) $\quad \forall\, x, y \in C_\varrho : p(x + y) = p(x) + p(y).$

p is an *(m)-norm*, if p is *directed over* C_ϱ, i. e. if

$$\text{(m)} \quad \forall\, x, y \in C_\varrho, \gamma \in H : (p(x) \leq \gamma \wedge p(y) \leq \gamma) \Rightarrow \\ \exists\, z \in C_\varrho : x \leq z \wedge y \leq z \wedge p(z) \leq \gamma.$$

Both properties of the norm p are already determined by the restriction of p onto C_ϱ. Let p be an (l)-norm. The restriction of p onto the generating cone C_ϱ can be linearly extended to V in a unique manner. This extension is called the linear mapping *associated* with p and is denoted by B_p. We have therefore $\forall\, x \in C_\varrho : B_p(x) = p(x)$. We show that

$$(3.1) \quad \forall\, x \in V : B_p(x) \leq p(x).$$

⌈Let $x \in V$. Since $C_\varrho - C_\varrho = V$, we have $x = y - z$ with certain $y, z \in C_\varrho$, hence $(x + y + z) = 2y$. Consequently $2 \cdot B_p(y) = p(x + y + z) \leq p(x) + p(y + z) = p(x) + B_p(y + z) = p(x) + 2 \cdot B_p(y) - B_p(x)$, from which we obtain $B_p(x) \leq p(x)$.⌋

An (l)-norm $p : V \to H$ is always *symmetric-monotone*, i. e.

$$(3.2) \quad \forall\, x, y \in V : -y \leq x \leq y \Rightarrow p(x) \leq p(y).$$

⌈Let $x, y \in V$, $-y \leq x \leq y$. Then we have $0 \leq x + y \leq 2y$ and $0 \leq -x + y \leq 2y$, consequently $p(2y) = p(x + y - x + y) = p(x + y) + p(-x + y)$. On the other hand $p(2x) = p(x + y + x - y) \leq p(x + y) + p(x - y) = p(x + y) + p(y - x) = p(2y)$. Hence $p(x) \leq p(y)$.⌋

In those cases we mainly deal with in this paper the (m)-norms have a somewhat more restricted property than is expressed by (m), because they are 'directed'. This will be shown for the absolute-monotone (m)-norms treated in § 5. Here we call a mapping ν of V into H *directed*, if $\forall\, x, y \in V, \gamma \in H$: $(\nu(x) \leq \gamma \wedge \nu(y) \leq \gamma) \Rightarrow \exists\, z \in V : x \leq z \wedge y \leq z \wedge \nu(z) \leq \gamma$. If this property holds only for all $x, y \in C_\varrho$, then ν is called *directed over* C_ϱ, as was already done for a norm p in definition (m).

Directed are also those (m)-norms p, whose norm balls $K_p[\gamma] = \{x \in V : p(x) \leq \gamma\}$ with $\gamma \in C_\sigma$ contain each a greatest element $e(\gamma)$. These norms we call (m)-norms *with maxima*. Clearly, to each (m)-norm with maxima there corresponds a mapping e_p of C_σ in C_p, the *associated mapping* e_p, such that $K_p[\gamma] \subset [-e_p(\gamma), e_p(\gamma)]$ and $e_p(\gamma) \in K_p[\gamma]$ for all $\gamma \in C_\sigma$. For e_p and all $x \in V$, $\gamma \in C_\sigma$ we have

(3.3) $\quad -x, x \leq e_p(p(x))$ and $p(e_p(\gamma)) \leq \gamma$.

If e_p is linear, then the unique linear extension of e_p onto H is also called the associated mapping.

Properties of mappings ν into ordered vector spaces can frequently be characterized by properties of their inverse predomains $K_\nu[\gamma] = \{x \in V : \nu(x) \leq \gamma\}$, $\gamma \in H$. An example thereof is the following characterization of the directedness of a mapping. The proof follows immediately from the definitions.

(3.4) **Theorem.** Let ν be a mapping of V into H. Then we have

ν is directed $\Leftrightarrow \forall \gamma \in H : K_\nu[\gamma]$ is directed.

Now we come to the problem of dualizing (l)-norms and (m)-norms. Let p be an H-valued norm on V. For any $f \in V^*$ let $B_p[f]$ be the set of all linear upper bounds of f from C_σ^0. Conversely, starting from a linear functional $\gamma' \in C_\sigma^0$ one can consider the set of all linear functionals $f \in V^*$, which are bounded by γ'. I. e., the relation 'γ' is an upper bound of f' generates not only the family B_p but also a family of subsets of V^*. We define for $\gamma' \in C_\sigma^0$ $K_p^{\mathrm{d}}[\gamma'] := \{f \in V^* : f(x) \leq \gamma' \cdot p(x), \forall x \in V\}$ and call K_p^{d} the *dual family* of the family K_p of norm balls of p. We have $f \in K_p^{\mathrm{d}}[\gamma'] \Leftrightarrow \gamma' \in B_p[f]$. For proving properties of $K_p^{\mathrm{d}}[\gamma']$ it is of importance that $K_p^{\mathrm{d}}[\gamma']$ is the polar set of the unit ball of the seminorm $\gamma' \cdot p$, i. e., $K_p^{\mathrm{d}}[\gamma']$ is equal to the unit ball of $(\gamma' \cdot p)^{\mathrm{d}}$, since we have for $\gamma' \in C_\sigma^0 : f \in K_p^{\mathrm{d}}[\gamma'] \Leftrightarrow (\forall x \in V : f(x) \leq \gamma' \cdot p(x)) \Leftrightarrow \forall x \in V : (\gamma' \cdot p(x) \leq 1 \Rightarrow f(x) \leq 1)$. For example one has immediately that $K_p^{\mathrm{d}}[\gamma']$ is convex and symmetric.

(3.5) **Lemma.** Let $p : V \to H$ be an (l)-norm and B_p its associated linear mapping, i. e. $B_p(x) = p(x)$, $\forall\, x \in C_\varrho$. Then we have for all $\gamma' \in C^0_\sigma$: $K^d_p[\gamma']$ contains $B^T_p(\gamma') = \gamma' \cdot B_p$ as its greatest element; in particular $K^d_p[\gamma']$ is directed.

⌈$f \in K^d_p[\gamma']$ means $f(x) \leq \gamma' \cdot p(x)$, $\forall\, x \in V$. Therefore $f \leq \gamma' \cdot B_p$ in the dual ordering in V^*. On the other hand from 3.1 follows $\gamma' \cdot B_p(x) \leq \gamma' \cdot p(x)$, $\forall\, x \in V$, i. e., $\gamma' \cdot B_p \in K^d_p[\gamma']$.⌋

(3.6) **Lemma.** Let $p : V \to H$ be a directed (m)-norm. Then for all $\gamma \in C_\sigma$ and $f, g \in V^b \cap C^0_\varrho$

$$\sup f(K_p[\gamma]) + \sup g(K_p[\gamma]) = \sup (f + g)\,(K_p[\gamma]).$$

⌈$\sup f(K_p[\gamma]) + \sup g(K_p[\gamma]) = \sup\,(f(K_p[\gamma]) + g(K_p[\gamma])) = \sup \{f(x) + g(y) : x \in K_p[\gamma],\ y \in K_p[\gamma]\} \geq \sup \{(f + g)\,(x) : x \in K_p[\gamma]\}$. For the proof of the converse relation let $x, y \in K_p[\gamma]$. Because of the directedness of p there exists a $z \in K_p[\gamma]$ with $x \leq z$ and $y \leq z$. Since $f, g \geq 0$, we have $f(x) + g(y) \leq f(z) + g(z) = (f + g)\,(z)$.⌋

In the case of (m)-norms with maxima and an associated mapping which is even linear the following Lemma holds.

(3.7) **Lemma.** Let $p : V \to H$ be an (m)-norm with maxima and a linear associated mapping $E_p : H \to V$. Then for all $\gamma' \in C^0_\sigma$, $f \in C^0_\varrho$

$$f \in K^d_p[\gamma'] \Leftrightarrow E^T_p(f) \leq \gamma', \text{ i. e., } f \cdot E_p(\gamma) \leq \gamma'(\gamma),\ \forall\, \gamma \in C_\sigma.$$

⌈$>$: Let $\gamma \in C_\sigma$. Because of 3.3, $p(E_p(\gamma)) \leq \gamma$, therefore $f(E_p(\gamma)) \leq \gamma' \cdot p(E_p(\gamma)) \leq \gamma'(\gamma)$. $<$: Let $x \in V$. From 3.3 follows $f(x) \leq f(E_p \cdot p(x)) \leq \gamma' \cdot p(x)$.⌋

The following propositions follow essentially from the Lemmas 3.5 and 3.7. They describe the relation between the order dual space $V^0 := C^0_\varrho - C^0_\varrho$ of (V, ϱ) and the space V^b of linear functionals which are bounded with respect to an (l)-norm or an (m)-norm.

(3.8) **Lemma.** Let p be an (H)-valued norm on V. Then we have

1. p is an (l)-norm $> V^{\mathrm{b}} \subset V^0$ and V^{b} is directed.

2. $(\forall x \in V : \exists z \in C_\varrho : x \leq z \wedge p(z) \leq p(x))$[1] $> V^{\mathrm{b}}$ is order-convex with respect to the dual ordering of V^*.

3. p is an (m)-norm with maxima and a linear associated mapping $> V^0 \subset V^{\mathrm{b}}$.

[1.: $f \in K_p^{\mathrm{d}}[\gamma'] > f \in [-\gamma' \cdot B_p, \gamma' \cdot B_p]$ and $\gamma' \cdot B_p \in K_p^{\mathrm{d}}[\gamma']$. 2.: Let $g, h \in V^{\mathrm{b}}$, i. e., $g(x) \leq \gamma' \cdot p(x)$, $h(x) \leq \beta' \cdot p(x)$ for all $x \in V$ and certain $\gamma', \beta' \in C_\sigma^0$. Let $g \leq f \leq h$. For all $x \in V$ we have with appropriate $z \in C_\varrho$ $x \leq z$ and $(h-f)(x) \leq (h-f)(z) \leq (h-g)(z) \leq (\gamma' + \beta') \cdot p(z) \leq (\gamma' + \beta') \cdot p(x)$. We obtain $h-f \in V^{\mathrm{b}}$. Analogously, $f-g \in V^{\mathrm{b}}$, hence $f \in V^{\mathrm{b}}$. 3. follows immediately from Lemma 3.7.]

Note that the 2. proposition of Lemma 3.8 is applicable in particular to directed (m)-norms.
Let $p : V \to H$ be a norm. If p is regular and has a dual norm p^{d}, then the relation between the dual family K_p^{d} of p and the family $K_{p^{\mathrm{d}}}$ of the norm balls of p^{d} is very close. Namely they are equal for all $\gamma' \in C_\sigma^0$, i. e.

$$(3.9) \quad \forall \gamma' \in C_\sigma^0 : K_p^{\mathrm{d}}[\gamma'] = K_{p^{\mathrm{d}}}[\gamma'],$$

as one can see from the definitions. Because of this coincidence the following theorem is obtained from Lemma 3.5.

(3.10) **Theorem.** Let $p : V \to H$ be a regular (l)-norm with an associated linear mapping $B_p : V \to H$. Then the dual norm p^{d} is an (m)-norm with a linear associated mapping, which is identical with $B_p^{\mathrm{T}} : H^0 \to V^{\mathrm{b}}$.

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[1] We call a norm with this property a *positive-directed* norm.

Analogously we obtain from Lemma 3.7 the corresponding theorem for (m)-norms which have a linear associated mapping.

(3.11) **Theorem.** Let $p : V \to H$ be a regular (m)-norm with maxima and a linear associated mapping $E_p : H \to V$. Then the dual norm p^d of p is an (l)-norm with $E_p^T : V^b \to H^0$ as its associated linear mapping.

Also in the general case of directed (m)-norms we get an analogous theorem. Here we require that p is completely regular instead of being only regular and therefore we can apply Lemma 3.6.

(3.12) **Theorem.** Let $p : V \to H$ be a completely regular directed (m)-norm. Then the dual norm p^d is an (l)-norm.

§ 4. Absolute-monotone norms.

Absolute-monotone norms are a generalization of the absolute and monotone norms on vector lattices [1]. We prove a duality theorem, which is essential for the proof of the duality theorems in § 5. The specification of the results to the case of norms on vector lattices is accomplished in § 8.

Let (V, ϱ) and (H, σ) be ordered vector spaces over R and v a mapping of V into H. v is called *symmetric-monotone*, if $\forall\, x, y \in V : -y \leq x \leq y \Rightarrow v(x) \leq v(y)$. A symmetric-monotone mapping v is also *monotone over* C_ϱ, i. e., $\forall\, x, y \in C_\varrho :$ $x \leq y \Rightarrow v(x) \leq v(y)$. v is called *symmetric-directed*, if $\forall\, x \in V$, $\gamma \in H : (v(x) \leq \gamma \wedge v(-x) \leq \gamma) \Rightarrow \exists\, y \in V : x \leq y \wedge -x \leq y \wedge$ $v(y) \leq \gamma$. If v is symmetric-monotone, symmetric-directed and symmetric, then we define v to be *absolute-monotone*.
A symmetric mapping v is symmetric-directed, if and only if $\forall\, x \in V : \exists\, y \in V : x \leq y \wedge -x \leq y \wedge v(y) \leq v(x)$. Therefrom follows immediately property (i) in the following

(4.1) **Lemma.** v is absolute-monotone $\succ$

(i) $\forall\, x \in V : \exists\, y \in V : x \in [-y, y] \wedge \nu(x) = \nu(y)$,

(ii) $\forall\, x \in V : \nu(0) \leq \nu(x)$.

⌈(ii): Let $x \in V$. Because of (i) $x \in [-y, y]$ and $\nu(x) = \nu(y)$ for a certain $y \in V$. Consequently $0 \leq y$, hence $0 \in [-y, y]$ and therefrom $\nu(0) \leq \nu(y) = \nu(x)$.⌋

Because of (ii) an absolute-monotone and positive-homogeneous mapping ν is semipositive, i. e. $0 \leq \nu(x)$, $\forall\, x \in V$. From (i) follows the important fact that the correspondence between absolute-monotone mappings and their restriction onto the cone C_ϱ is one-to-one. Let ν be a mapping of V into H. We denote the restriction of ν onto C_ϱ with ν_{C_ϱ} and call it the *generator* of ν. Then we have

(4.2) **Lemma.** ν is absolute-monotone $>$ $\forall\, x \in V$:

$$\nu(x) = \inf\{\nu_{C_\varrho}(y) : -y \leq x \leq y\} = \min\{\nu_{C_\varrho}(y) : -y \leq x \leq y\}.$$

Absolute-monotone norms p satisfy the following equivalent properties.

(i) $p(x) = \inf\{p(y) : -x, x \leq y\}$,

(ii) $p(x) = \inf\{p(y + z) : x = y - z,\ y, z \in C_\varrho\}$.

Because of the one-to-one correspondence of absolute-monotone mappings and their generators it is meaningful to ask for properties of ν and ν_{C_ϱ} which likewise correspond to one another. As an example thereof we note the following

(4.3) **Theorem.** Let $\nu : V \to H$ be absolute-monotone. Then

ν is sublinear $\gtrless$ ν_{C_ϱ} is sublinear.

⌈$<$: Let $x, y \in V$. $\nu(x) + \nu(y) = \inf\{\nu_{C_\varrho}(\bar{x}) : -\bar{x} \leq x \leq \bar{x}\}$ $+ \inf\{\nu_{C_\varrho}(\bar{y}) : -\bar{y} \leq y \leq \bar{y}\} = \inf\{\nu_{C_\varrho}(\bar{x}) + \nu_{C_\varrho}(\bar{y}) : -\bar{x} \leq x$ $\leq \bar{x}, -\bar{y} \leq y \leq \bar{y}\}$. Because of $[-\bar{x}, \bar{x}] + [-\bar{y}, \bar{y}] \subset [-\bar{x} - \bar{y},$

$\bar{x}+\bar{y}]$ follows $\nu(x+y) = \inf\{\nu_{C_\varrho}(\bar{z}): -\bar{z} \leq x+y \leq \bar{z}\} \leq \nu(x) + \nu(y)$. Similarly we obtain $\nu(\lambda x) = \lambda\nu(x)$, $\forall\, x \in V$, $\lambda \geq 0$.⌋

In the following theorem the properties introduced above are characterized by properties of the inverse predomains with respect to ν [11] $K_\nu[\gamma] = \{x \in V: \nu(x) \leq \gamma\}$, $\forall\, \gamma \in H$, i. e., in the case of norms, by properties of their norm balls. The proofs result directly from the definitions.

(4.4) **Theorem.** Let ν be a mapping of V into H. Then

1. ν is symmetric-directed $\Leftrightarrow$ $\forall\, \gamma \in H: K_\nu[\gamma]$ is *symmetric-directed*,[1] i. e. $x \in K_\nu[\gamma] \wedge -x \in K_\nu[\gamma] \Rightarrow \exists\, y \in K_\nu[\gamma]: x \leq y \wedge -x \leq y$.

2. ν is symmetric-monotone $\Leftrightarrow$ $\forall\, \gamma \in H: K_\nu[\gamma]$ is *symmetric-order-convex*,[1] i. e. $x \in K_\nu[\gamma] \wedge -x \in K_\nu[\gamma] \Rightarrow [-x, x] \subset K_\nu[\gamma]$.

3. ν is absolute-monotone $\Leftrightarrow$ $\forall\, \gamma \in H: K_\nu[\gamma]$ is *absolute-order-convex*,[1] i. e. symmetric-order-convex, symmetric-directed and symmetric.

4. ν is absolute-monotone $\Leftrightarrow$ $\forall\, \gamma \in H: K_\nu[\gamma] = \bigcup\{[-x, x]: \nu(x) \leq \gamma\}$ $\Leftrightarrow$ $\forall\, x \in V: K_\nu[\nu(x)] = \bigcup\{[-y, y]: y \in \nu^{-1}\{\nu(x)\}\}$.

The remainder of § 4 aims at the duality theorem 4.7 for absolute-monotone norms.

Let p be an H-valued norm on V and K_p^{d} the dual family of the family of norm balls of p, i.e. $K_p^{\mathrm{d}}[\gamma'] = \{f \in V^*: f(x) \leq \gamma' \cdot p(x), \forall\, x \in V\}$ for $\gamma' \in C_\sigma^0$.

(4.5) **Lemma.** Let $\gamma' \in C_\sigma^0$. We have

1. p is symmetric-directed $>$ $K_p^{\mathrm{d}}[\gamma']$ is symmetric-order-convex.

2. p is symmetric-monotone $>$ $K_p^{\mathrm{d}}[\gamma']$ is symmetric-directed.

3. p is absolute-monotone $>$ $K_p^{\mathrm{d}}[\gamma']$ is absolute-order-convex.

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[1] See also [2].

⌈1. Let $g \in [-f, f]$ and $-f, f \in K_p^d[\gamma']$. For all $x \in V$ there exists a $y \in V$ such that $x \leq y$, $-x \leq y$ and $p(y) \leq p(x)$. We have $g(x) \leq f(y) \leq \gamma' \cdot p(y) \leq \gamma' \cdot p(x)$, therefore $g \in K_p^d[\gamma']$.
2. Let $-f, f \in K_p^d[\gamma']$. Since f is bounded, there exists $q(x) := \sup f([-x, x])$ for all $x \in C_\varrho$. We have $q(x) \leq \gamma' \cdot p(x)$, $\forall\, x \in C_\varrho$. Since q is superlinear and $\gamma' \cdot p$ is a seminorm on V, according to a theorem of Bonsall (see [14] p. 13) there exists a linear functional g such that $q(x) \leq g(x)\ \forall\, x \in C_\varrho$ and $g(x) \leq \gamma' \cdot p(x)$ $\forall\, x \in V$. From the definition of q follows $f \leq g$ and $-f \leq g$. On the other hand we also have $g \in K_p^d[\gamma']$.⌋

We remark that in the proof of 2. and thereby 3. the theorem of Hahn-Banach is used in the form of a theorem of Bonsall [8]. This is the only place in the present paper in which the theorem of Hahn-Banach is directly used.

(4.6) **Corollary.** Let $p : V \to H$ be an absolute-monotone norm. Then V^b is an ideal in V^0, i. e., V^b is an order-convex and directed subspace of V^0.

Now let $p : V \to H$ be a regular norm and p^d its dual. In 3.9 we have mentioned that for all $\gamma' \in C_\sigma^0$ $K_p^d[\gamma'] = K_{p^d}[\gamma']$. Because of this equality we obtain from theorem 4.4 together with lemma 4.5 the following duality theorem.

(4.7) **Theorem.** Let $p : V \to H$ be a regular absolute-monotone norm. Then the dual norm p^d is also absolute-monotone.

§ 5. (al)-norms and (am)-norms.

If an (l)-norm is absolute-monotone, then we call it an *(al)-norm*. An absolute-monotone (m)-norm is called an *(am)-norm*. The duality theorems for (al)- and (am)-norms result from the combination of the corresponding theorems in §§ 3 and 4, exactly as it happens for the definition of (al)- and (am)-norms. Before we write down these theorems let us investigate special properties of (al)- and (am)-norms. If the ordered vector spaces underlying the (al)- and (am)-norms are even vector lattices, then these norms coincide with the classical (AL)- and (AM)-norms, respectively. In the following let C_ϱ and C_σ be generating again.

Since an absolute-monotone norm p is determined through its one-to-one correspondence with its generator, an (al)-norm is determined through its one-to-one correspondence with its associated linear mapping B_p.

(5.1) **Lemma.** Let p be an (al)-norm and B_p its associated linear mapping. Then we have the following 'interpolation property'.

$\forall\, x, y, u, v \in V : x, y \leq u, v \Rightarrow \exists\, z \in V : x, y \leq z \wedge B_p(z) \leq B_p(u), B_p(v)$.

⌈For $y: = -x$ there is a z such that $p(x) = p(y) = p(z)$ and $x, y \leq z$. Since p is symmetric-monotone, we have also $p(x) \leq p(u)$ and $p(x) \leq p(v)$. p can be substituted with B_p, because $u, v, z \in C_\varrho$. The general case, y being arbitrary, can be obtained via the transformation $x' = x - \frac{1}{2}(x + y)$, $y' = y - \frac{1}{2}(x + y)$ from the special case.⌋

In the following we introduce the concept of an 'order-convex' mapping and show in corollary 5.4 that an (am)-norm is order-convex and directed. To the fore we remark that every directed mapping is symmetric-directed and directed over C_ϱ. For symmetric mappings the converse is also true. The proof is simple.

(5.2) Let ν be symmetric. Then we have
ν is directed $\Leftrightarrow$ ν is symmetric-directed and directed over C_ϱ.

(5.3) **Lemma.** Let $\nu : V \to H$ be absolute-monotone and directed over C_ϱ. Then

(i) ν is *order-convex*, i. e. $\forall\, x, y, z \in V, \gamma \in H$:
$(y \leq x \leq z \wedge \nu(y) \leq \gamma \wedge \nu(z) \leq \gamma) \Rightarrow \nu(x) \leq \gamma$.

(ii) $\forall\, x, y \in V, \gamma \in H: (\nu(x) \leq \gamma \wedge \nu(y) \leq \gamma) \Rightarrow$
$\exists\, z \in C_\varrho: x, y \in [-z, z] \wedge \nu(x), \nu(y) \leq \nu(z) \leq \gamma$;
in particular, ν is directed.

⌈(i): First of all, since ν is absolute-monotone, the premise implies $-\bar{y} \leq y \leq \bar{y}$, $-\bar{z} \leq z \leq \bar{z}$ and $\nu(\bar{y}) \leq \gamma$, $\nu(\bar{z}) \leq \gamma$ for certain $\bar{y}, \bar{z} \in C_\varrho$. Because of the directedness of ν, as noted in (5.2), there also is an $\bar{x}$ with $\bar{y}, \bar{z} \leq \bar{x}$ and $\nu(\bar{x}) \leq \gamma$. Because of $-\bar{x} \leq y \leq x \leq z \leq \bar{x}$, i. e. $-\bar{x} \leq x \leq \bar{x}$, there follows $\nu(x) \leq \nu(\bar{x}) \leq \gamma$.
(ii): From lemma 4.1 we get $\nu(\bar{x}) = \nu(x)$, $x \in [-\bar{x}, \bar{x}]$, $\nu(y) = \nu(\bar{y})$, $y \in [-\bar{y}, \bar{y}]$ for certain $\bar{x}, \bar{y} \in C_\varrho$. Since ν is directed over C_ϱ, we obtain $\bar{x}, \bar{y} \leq z$, $\nu(z) \leq \gamma$ for a certain $z \in C_\varrho$. Because of $x, y \in [-z, z]$ we conclude $\nu(x), \nu(y) \leq \nu(z) \leq \gamma$.⌋

(5.4) **Corollary.** Let $p : V \to H$ be a norm. Then

p is an absolute-monotone (m)-norm $\Leftrightarrow$ p is directed and order-convex.

The property of a mapping $\nu : V \to H$ to be order-convex may be characterized by properties of the inverse predomains $K_\nu[\gamma] = \{x \in V : \nu(x) \leq \gamma\}$, $\gamma \in H$ as it is done in the theorems 3.4 and 4.4.

(5.5) **Theorem.** Let ν be a mapping of V into H. Then

ν is order-convex $\Leftrightarrow$ $\forall\, \gamma \in H: K_\nu[\gamma]$ is order-convex.

If the norm balls $K_p[\gamma]$ of an (am)-norm p contain a greatest element $e_p(\gamma)$ for all $\gamma \in C_\sigma$, then, because $K_p[\gamma]$ is order-convex, $K_p[\gamma]$ is equal to the interval $[-e_p(\gamma), e_p(\gamma)]$. In such a case p is called an 'order-interval' norm. I. e. p is an *order-interval norm*, if $\forall\, \gamma \in C_\sigma : \exists\, y \in V : K_p[\gamma] = [-y, y]$. Because of corollary 5.4 together with theorem 3.4 and 5.5 we have

(5.6) p is an order-interval norm $\Leftrightarrow$ p is an (am)-norm with maxima.

In the following we state the duality theorems for (al)- and (am)-norms. They arise essentially as corollaries of the corresponding theorems in §§ 3, 4. First of all we consider the dual family K_p^{d} of an (al)-norm and an (am)-norm and in particular the relation between V^{b} and V^0.

(5.7) **Lemma.** Let $p : V \to H$ be a norm. Then

1. p is an (al)-norm with the associated linear mapping $B_p : V \to H > \forall\, \gamma' \in C_\sigma^0 : K_p^{\mathrm{d}}[\gamma'] = [-B_p^{\mathrm{T}}(\gamma'), B_p^{\mathrm{T}}(\gamma')]$.

2. p is an order-interval norm with a linear associated mapping $E_p : H \to V > \forall\, \gamma' \in C_0^\sigma : K_p^{\mathrm{d}}[\gamma'] = \bigcup \{[-f, f] : E_p^{\mathrm{T}}(f) \leq \gamma'\}$.

⌈1. Lemma 3.5 together with Lemma 4.5 (1.).
2. Lemma 3.7 together with Lemma 4.5 (3.).⌋

It should be noted that for the proof of the 2. proposition of the lemma above the theorem of Hahn-Banach is needed indirectly via lemma 4.5 (2.). We obtain the following

(5.8) **Corollary.** Let $p : V \to H$ be a norm. Then

1. p is an (al)-norm $>$ V^{b} is an ideal in V^0.

2. p is an order-interval norm with a linear associated mapping $> V^{\mathrm{b}} = V^0$.

(5.9) **Theorem.** Let $p : V \to H$ be a regular (al)-norm with the associated linear mapping $B_p : V \to H$. Then the dual norm p^{d} of p is an order-interval norm, which has the linear mapping $B_p^{\mathrm{T}} : H^0 \to V^{\mathrm{b}}$ as its associated mapping.

⌈Theorem 3.10, Lemma 5.7. (1.)⌋

(5.10) **Theorem.** Let $p : V \to H$ be a regular order-interval norm with a linear associated mapping $E_p : H \to V$. Then the dual norm p^{d} is an (al)-norm with the associated linear mapping $E_p^{\mathrm{T}} : V^{\mathrm{b}} \to H^0$.

[Theorem 3.11 and theorem 4.7.]

(5.11) **Theorem.** Let $p : V \to H$ be a completely regular (am)-norm. Then the dual norm p^{d} is an (al)-norm.

[Theorem 3.12 and 4.7.]

§ 6. Order-interval norms and absolute value operators over ordered vector spaces.

At the beginning of the foregoing paragraph we have claimed the consistency between the concepts of (al)- and (am)-norms and the classical concepts of (AL)- and (AM)-norms on vector lattices. This will be shown in § 8. But the definitions turn out to be also adequate, because many theorems about (AL)-norms and (AM)-norms on vector lattices can be generalized to (al)- and (am)-norms. The parallels of statements for absolute norms on vector lattices to the one and absolute-monotone norms on ordered vector spaces to the other seem to be most far-reaching in the case of orderinterval norms because these latter norms induce an operation onto their preimage space which we may think of as the formation of an absolute value. In this paragraph we shall investigate the relation between order-interval norms p and generalized absolute value operators or, equivalently, sublattice structures in the ordered preimage space (V, ϱ) of p. A few of the results presently obtained shall be needed in § 7 for the proof of necessary and sufficient regularity conditions for (al)-norms.

Let (V, ϱ) and (H, σ) be ordered vector spaces. As we have already defined an H-valued norm p on V is an order-interval norm, if

$\forall\, \gamma \in C_\sigma : \exists\, y \in C_\varrho : K_p[\gamma] = \{x \in V : p(x) \leq \gamma\} = [-y, y]$.

The associated mapping of an order-interval norm is a so-called 'absolute-inverse' of the norm. Here we call a mapping $e : C_\sigma \to C_\varrho$ an *absolute-inverse* of a mapping $v : V \to C_\sigma$, if

(6.1) $\forall \gamma \in C_\sigma : K_\nu[\gamma] = \{x \in V : \nu(x) \leq \gamma\} = [-e(\gamma), e(\gamma)]$

or, equivalently,

(6.2) $\forall x \in V, \gamma \in C_\sigma : \nu(x) \leq \gamma \Leftrightarrow -x, x \leq e(\gamma)$.

If for a mapping an absolute-inverse exists, then it is uniquely determined. Conversely, for any mapping $e : C_\sigma \to C_\varrho$ there exists at most one mapping $\nu : V \to C_\sigma$ such that 6.1 or 6.2 holds. This is a consequence of the following equations for mappings $\nu : V \to C_\sigma$ with an absolute-inverse $e : C_\sigma \to C_\varrho$.

(6.3) $\nu(x) = \inf \{\gamma \in C_\sigma : x \in [-e(\gamma), e(\gamma)]\}, \forall x \in V$ and

(6.4) $e(\gamma) = \sup \{x \in V : \nu(x) \leq \gamma\} = \sup K_\nu[\gamma]$.

From 6.1 there follows immediately that a norm $p : V \to H$ is an order-interval norm if and only if it has an absolute-inverse.

A pair of mappings consisting of an order-interval norm p and its absolute-inverse e possesses properties which are analogous to a large extent to those of a Galois connection (lemma 6.5). We have $p \cdot e \cdot p = p$, $e \cdot p \cdot e = e$, and $p \cdot e$ is a hull operator. In order to characterize the operator $e \cdot p$ we introduce the concept of an 'absolute value operator' over an ordered vector space. Then an order-interval norm proves to be decomposable into an absolute value operator and an isomorphism between subsets of V and H.

A mapping h of an ordered vector space (V, ϱ) into itself is called an *absolute value operator* over (V, ϱ), if

(i) h is *symmetric-extensive*, i.e. $\forall x \in V : -x, x \leq h(x)$,

(ii) h is symmetric-monotone, i.e. $\forall x, y \in V : -y \leq x \leq y \Rightarrow h(x) \leq h(y)$,

(iii) h is idempotent, i.e. $h^2 = h$.

The image $h(V)$ of V with respect to h is equal to the set of invariant elements with respect to h and is called the *kernel* of h.

(6.5) **Lemma.** Let $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$. Then all the properties (ii)–(x) follow from (i).

(i) e is the absolute-inverse of ν, i.e. $\forall\, x \in V, \gamma \in C_\sigma$
$\nu(x) \leq \gamma \Leftrightarrow -x, x \leq e(\gamma)$.

(ii) $e \cdot \nu$ is symmetric-extensive, i.e. $\forall\, x \in V$
$-x, x \leq e \cdot \nu(x)$.

(iii) $\nu \cdot e$ is extensive with respect to $\leq_\sigma^{-1}$, i.e. $\forall\, \gamma \in C_\sigma$
$\nu \cdot e(\gamma) \leq \gamma$.

(iv) ν is symmetric-monotone, i.e. $\forall\, x, y \in V$
$-y \leq x \leq y \Rightarrow \nu(x) \leq \nu(y)$.

(v) e is monotone over C_σ, i.e. $\forall\, \gamma, \beta \in C_\sigma$
$\gamma \leq \beta \Rightarrow e(\gamma) \leq e(\beta)$.

(vi) $\nu \cdot e \cdot \nu = \nu$ and $e \cdot \nu \cdot e = e$.

(vii) $e \cdot \nu$ is an absolute value operator over (V, ϱ) with kernel $e(C_\sigma)$.

(viii) $\nu \cdot e$ is a hull operator over $(C_\sigma, \leq_\sigma^{-1})$ with kernel $\nu(V)$.

(ix) The restricted mappings $\nu|_{e(C_\sigma)}$ and $e|_{\nu(V)}$ are the inverse mappings of each other.

(x) $\nu|_{e(C_\sigma)}$ and $e|_{\nu(V)}$ are order isomorphisms of $e(C_\sigma)$ onto $\nu(V)$ and of $\nu(V)$ onto $e(C_\sigma)$, respectively, each with respect to the orderings induced by ϱ and σ.

⌈(ii): $\nu(x) \leq \nu(x) \Rightarrow -x, x \leq e(\nu(x))$.
(iii): $0 \leq e(\gamma) \leq e(\gamma) \Rightarrow -e(\gamma), e(\gamma) \leq e(\gamma) \Rightarrow \nu(e(\gamma)) \leq \gamma$.
(iv): $-y, y \leq e(\nu(y)) \Rightarrow -x, x \leq e(\nu(y)) \Rightarrow \nu(x) \leq \nu(y)$.
(v): $\nu(e(\gamma)) \leq \gamma \Rightarrow \nu(e(\gamma)) \leq \beta \Rightarrow e(\gamma) \leq e(\beta)$.
(vi): From (iii) it follows that $\nu \cdot e(\nu(x)) \leq \nu(x)$. On the other hand from $-x, x \leq e \cdot \nu(x)$ follows with (iv) $\nu(x) \leq \nu(e \cdot \nu(x))$. Hence $\nu \cdot e \cdot \nu = \nu$. Correspondingly, we have $-e(\gamma) \leq 0 \leq e(\gamma) \leq e \cdot \nu(e(\gamma))$. On the other hand from $\nu \cdot e(\gamma) \leq \gamma$ follows with (v) $e(\nu \cdot e(\gamma)) \leq e(\gamma)$. Therefore $e \cdot \nu \cdot e = e$.
(vii)–(x) are an immediate consequence of (ii)–(vi).⌋

The foregoing lemma suggests different ways to express when a mapping e is the absolute-inverse of a mapping v.

(6.6) **Theorem.** Let $v: V \to C_\sigma$ and $e: C_\sigma \to C_\varrho$. Then we have the following relations between the properties (i)–(x) in lemma 6.5.

1. (i) $\gtrless$ (ii)–(v).

2. (i) $\gtrless$ (vii)–(x).

⌈1. $<$: Let $v(x) \leq y$. We obtain $e(v(x)) \leq e(y)$. Because of $-x, x \leq e \cdot v(x)$ we have $-x, x \leq e(y)$. Conversely, let $-x, x \leq e(y)$. We obtain $-e(y) \leq x \leq e(y)$. Therefrom $v(x) \leq v(e(y))$. Because of $v \cdot e(y) \leq y$, we have $v(x) \leq y$.
2. $<$: One has to prove (ii)–(v) and then conclude using 1. (ix) together with (vii) and (viii) imply (vi). (iv) holds, since $e \cdot v$ is symmetric-monotone and v is an order isomorphism onto $e(C_\sigma)$. (v) follows in the same way.⌋

Theorem 6.6 yields a construction of mappings with an absolute-inverse from operators over V and C_σ, resp., and an isomorphism of their kernels. This is done in the following corollary.

(6.7) **Corollary.** Let $h: V \to V$, $l: C_\sigma \to C_\sigma$ be operators, i a mapping of $h(V)$ in $l(C_\sigma)$. Then the following properties are equivalent.

(i) h is an absolute value operator over (V, ϱ), l a hull operator over C_σ with respect to $\leq_\sigma^{-1}$ and i an isomorphism of $h(V)$ onto $l(C_\sigma)$ with respect to the induced orderings.

(ii) $e := i^{-1} \cdot l$ is an absolute-inverse of $v := i \cdot h$.

The absolute-inverse e of a mapping v is gained from v through the relation 6.4. Conversely, v is gained from e through 6.3. A consequence of these equations is that there is a one-to-one correspondence between mappings and their absolute-inverse, as has already been noted. Because of this correspondence we can associate certain properties of v and e to one another.

(6.8) **Theorem.** Let $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$ be mappings such that e is the absolute-inverse of ν. Then we have the following equivalences.

1. ν is positive-definite $\gtrless$ $e(0) = 0$.

2. ν is positive-homogeneous $\gtrless$ e is positive-homogeneous.

3. ν is sublinear $\gtrless$ e is superlinear.

4. ν is an order-interval norm $\gtrless$ e is superlinear.

⌈1. $>$: $e(0) = \sup\{x \in V : \nu(x) \leq 0\} = 0$. $<$: $\nu(0) = \inf\{\gamma \in C_\sigma : 0 \leq e(\gamma)\} = 0$. $\nu(x) = 0 \Rightarrow -x, x \leq e(0) = 0 \Rightarrow x = 0$.
2. $>$: $\forall\, \lambda \geq 0, \gamma \in C_\sigma$ we have $\nu \cdot e(\lambda\gamma) \leq \lambda\gamma, \frac{1}{\lambda}\nu \cdot e(\lambda\gamma) \leq \gamma$, $\nu\left(\frac{1}{\lambda}e(\lambda\gamma)\right) \leq \gamma$, hence $\frac{1}{\lambda}e(\lambda\gamma) \leq e(\gamma)$ or $e(\lambda\gamma) \leq \lambda e(\gamma)$. The substitution $\beta = \lambda\gamma$, $\mu = \frac{1}{\lambda}$ yields $\mu e(\beta) \leq e(\mu\beta)$, $\forall\, \mu \geq 0$, $\beta \in C_\sigma$. $<$: Similarly.
3. $>$: $e(\gamma) + e(\beta) \leq e \cdot \nu\big(e(\gamma) + e(\beta)\big) \leq e\big(\nu(e(\gamma)) + \nu(e(\beta))\big) \leq e(\gamma + \beta)$. $<$: $-x, x \leq e \cdot \nu(x)$ and $-y, y \leq e \cdot \nu(y)$ imply $\nu(x + y) \leq \nu\big(e(\nu(x)) + e(\nu(y))\big)$. Because of $e\big(\nu(x)\big) + e\big(\nu(y)\big) \leq e\big(\nu(x) + \nu(y)\big)$ we obtain $\nu(x + y) \leq \nu \cdot e\big(\nu(x) + \nu(y)\big) \leq \nu(x) + \nu(y)$.
4.: If e is positive-homogeneous, then $2e(0) = e(0)$, i.e. $e(0) = 0$.⌋

Let $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$ be mappings such that e is the absolute-inverse of ν. We next show, that e maps an infimum of any subset of C_σ into an infimum of its image. Here we have to take the infima with respect to the relative orderings of C_σ and C_ϱ, respectively, which are induced by σ and ϱ, respectively. Let $A \subset C_\sigma$ and $x \in C_\sigma$. Then $x = \inf_{C_\sigma} A$ means that $x \leq a$, $\forall\, a \in A$ and $\forall\, y \in C_\sigma : (y \leq a, \forall\, a \in A) \Rightarrow y \leq x$. We deal analogously, whenever we consider infima or suprema relative to orderings restricted to subsets.

(6.9) **Lemma.** Let $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$ be mappings such that e is the absolute-inverse of ν. Then for $A \subset C_\varrho$, $B \subset C_\sigma$, $x \in C_\varrho$ and $\gamma \in C_\sigma$

1. $x = \sup A \Rightarrow \nu(x) = \sup \nu(A)$,

2. $\gamma = \inf_{C_\sigma} B \Rightarrow e(\gamma) = \inf_{C_\varrho} e(B)$.

[1. $\nu(a) \leq \nu(x)$, $\forall\, a \in A$ since ν is monotone over C_ϱ. Let $\gamma \in H$ and $\nu(a) \leq \gamma$, $\forall\, a \in A$. Then $-a, a \leq e(\gamma)$, $\forall\, a \in A$, hence $x \leq e(\gamma)$. Consequently, $-x, x \leq e(\gamma)$. Therefrom $\nu(x) \leq \nu(e(\gamma)) \leq \gamma$.
2. $e(\gamma) \leq e(\beta)$, $\forall\, \beta \in B$ since e is monotone over C_σ. Let $x \in C_\varrho$ and $x \leq e(\beta)$, $\forall\, \beta \in B$. We have $-x, x \leq e(\beta)$, hence $\nu(x) \leq \nu(e(\beta)) \leq \beta$ for all $\beta \in B$. Therefrom follows $\nu(x) \leq \gamma$. We obtain $x \leq e(\gamma)$.]

If (V, ϱ) is a vector lattice, then the infima taken relative to C_ϱ are also infima in (V, ϱ) and, accordingly, the lemma above may be extended. I. e. instead of 2. we have

2′. $\gamma = \inf_{C_\sigma} B \Rightarrow e(\gamma) = \inf e(B)$.

We note the following

(6.10) **Corollary.** Let (V, ϱ) and (H, σ) be vector lattices, $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$. Then

e is an absolute-inverse of ν $\succ$ e is a $\wedge$-homomorphism, i. e. $\forall\, \gamma, \beta \in C_\sigma : e(\gamma \wedge \beta) = e(\gamma) \wedge e(\beta)$.

Now, we leave the general part of the theory of order-interval norms and turn to those possessing an absolute-inverse which is even linear. To this class of norms belong all the completely regular order-interval norms.

(6.11) **Theorem.** Let $\nu : V \to C_\sigma$ be a mapping which possesses an absolute-inverse $e : C_\sigma \to C_\varrho$. Let $U := \text{Lin}\,(e \cdot \nu(V))$. We consider the following properties.

(i) e is linear.

(ii) $e(C_\sigma)$ is a convex cone with the property $C_\varrho \cap U = e(C_\sigma)$.

(iii) $\forall\, x \in V : \inf \{y \in U : -x, x \leq y\} = e \cdot \nu(x)$.

(iv) $\forall x \in U$:
$\inf\{y \in V: -x, x \leq y\} = \sup\{-x, x\} = e \cdot \nu(x)$.

(v) $\forall x \in V$:
$\inf\{y \in U: x \leq y\} = e \cdot \nu(x + e \cdot \nu(x)) - e \cdot \nu(x)$.

(vi) $\forall x \in V$:
$\sup\{y \in U: y \leq x\} = e \cdot \nu(x) - e \cdot \nu(-x + e \cdot \nu(x))$.

Then we have

1. (i) $>$ (ii)–(vi).

2. (ii) $\asymp$ (iii) $\asymp$ (iv).

3. (ii) $>$ (v) and (v) $\asymp$ (vi).

[(i) $>$ (ii): Clearly, $e(C_\sigma)$ is convex and we have $e(C_\sigma) \subset C_\varrho \cap U$. Let $x = e(\gamma) - e(\beta)$ and $x \in C_\varrho$. Since ν is monotone over C_ϱ, $\nu \cdot e(\beta) \leq \nu \cdot e(\gamma)$, hence $\nu \cdot e(\gamma) - \nu \cdot e(\beta) \in C_\sigma$. Because of (i), $e(\nu \cdot e(\gamma)) = e(\nu \cdot e(\gamma) - \nu \cdot e(\beta) + \nu \cdot e(\beta)) = e(\nu \cdot e(\gamma) - \nu \cdot e(\beta)) + e(\nu \cdot e(\beta))$, therefore $e(\gamma) - e(\beta) = e(\nu \cdot e(\gamma) - \nu \cdot e(\beta)) \in e(C_\sigma)$.
(ii) $>$ (iii): We have $-x, x \leq e \cdot \nu(x)$. Let $-x, x \leq y \in U$. We have $y \in C_\varrho$. Because of (ii), $y \in e(C_\sigma)$, i. e. $y = e(\gamma)$ for a $\gamma \in C_\sigma$. We have $-x, x \leq e(\gamma)$, hence $\nu(x) \leq \gamma$. Therefrom $e \cdot \nu(x) \leq e(\gamma) = y$.
(iii) $>$ (v): Because of $0 \leq x + e \cdot \nu(x)$ and (iii) we have $\inf\{y \in U: -x - e \cdot \nu(x).\ x + e \cdot \nu(x) \leq y\} = \inf\{y \in U: x + e \cdot \nu(x) \leq y\} = e \cdot \nu(x + e \cdot \nu(x))$. Since $e \cdot \nu(x) \in U$, we have also $\inf\{y \in U: x + e \cdot \nu(x) \leq y\} = e \cdot \nu(x) + \inf\{z \in U: x \leq z\}$. We obtain (v).
(v) $\asymp$ (vi): We have $e \cdot \nu(-x + e \cdot \nu(-x)) - e \cdot \nu(-x) = \inf\{y \in U: -x \leq y\} = -\sup\{z \in U: z \leq y\}$
$= e \cdot \nu(-x + e \cdot \nu(x)) - e \cdot \nu(x)$.
(iii) $>$ (iv): $-x, x \leq e \cdot \nu(x)$ since e is symmetric-extensive. Let $y \in V$ and $-x, x \leq y$. Because of (vi), $-x, x \leq e \cdot \nu(y) - e \cdot \nu(-y + e \cdot \nu(y)) \leq y$. Using (iii) we obtain $e \cdot \nu(x) \leq e \cdot \nu(y) - e \cdot \nu(-y + e \cdot \nu(y)) \leq y$.
(iv) $>$ (ii): Let $u, v \in e(C_\sigma) = e \cdot \nu(V)$, i.e. $u = e \cdot \nu(x)$, $v = e \cdot \nu(y)$ for certain $x, y \in V$. $-u - v \leq u + v$ since

$u, v \in C_\varrho$. Therefrom with (iv) we obtain $e \cdot \nu(u + v) = u + v$, i. e. $u + v \in e(C_\sigma)$. Now let $u \in C_\varrho \cap U$, i. e. $u = e \cdot \nu(x) - e \cdot \nu(y)$, $0 \leq u$. Because of (iv) there follows again $e \cdot \nu(u) = u \in e(C_\sigma)$.⌋

From property (iv) of the foregoing theorem follows that the subspace U is a sublattice of (V, ϱ). Property (v), moreover, says that U contains a least upper bound for each element $x \in V$. Therefore we define the notion of an 'order-projective subvector-lattice'. Let U be a subspace of an ordered vector space (V, ϱ). U is called a *subvector-lattice*, if $\forall\, x, y \in U : \exists\, z \in U :$ $z = \sup\,\{x, y\}$. Here $z = \sup\,\{x, y\}$ means that z is the supremum of x and y with respect to the ordering ϱ. U is called *order-projective in* (V, ϱ) if $\forall\, x \in V : \exists\, y \in U : y = \inf\,\{z \in U : x \leq z\}$. Each order-projective subspace of V is *cofinal*, i. e. $\forall\, x \in V : \exists\, y \in U : x \leq y$. Now from theorem 6.11 follows easily

(6.12) **Corollary.** Let $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$ be mappings such that e is the absolute-inverse of ν and let e be linear. Then for $U := \mathrm{Lin}\,(e \cdot \nu(V))$ we have

1. U is a subvector-lattice of (V, ϱ).

2. U is order-projective in (V, ϱ); in particular, U is *band-closed*, i. e. $\forall\, A \subset U, x \in V : x = \inf A \Rightarrow x \in U$.

⌈1.: From (iv) follows that $\sup\,\{-x, x\}$ exists for all $x \in U$, therefore $\sup\,\{2x, 0\}$ and $\sup\,\{x, 0\}$ exist for all $x \in U$. Since $\sup\,\{x, 0\} = \sup\left\{\frac{1}{2}\,x, -\frac{1}{2}\,x\right\} + \frac{1}{2}\,x \in U$, the conclusion follows.
2.: As a consequence of (v) U is order-projective. Now, let $A \subset U$, $x \in V$ and $x = \inf A$. Because of (v) we have also $x \leq e \cdot \nu\,(x + e \cdot \nu(x)) - e \cdot \nu(x) \leq a$, $\forall\, a \in A$. Therefrom we obtain $x = e \cdot \nu\,(x + e \cdot \nu(x)) - e \cdot \nu(x)$, hence $x \in U$.⌋

It is easily verified that the absolute value operation $|x| = x \vee -x$ in a vector lattice (V, ϱ) has the identical mapping of C_ϱ as its absolute-inverse. Moreover, we have that an ordered vector space (V, ϱ) is a vector lattice if a mapping $|.| : V \to C_\varrho$ of V onto C_ϱ exists such that the identical mapping of C_ϱ is the absolute-inverse of $|.|$. This result is a consequence of corollary 6.12, but, naturally, it is easier to prove it directly by a few simple arguments.

The statements of theorem 6.12 concern the subspace $U = \text{Lin}\,(e \cdot v(V))$ and the absolute value operator $e \cdot v$ over V. The following theorem summarizes properties of v, e and of the hull operator $v \cdot e$. Again it is supposed that e is linear.

(6.13) **Theorem.** Let $v : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$ be mappings, e the absolute-inverse of v and e linear. Let C_σ be generating, i. e. $C_\sigma - C_\sigma = H$. Then we have

1. v is a norm, namely an order-interval norm.

2. v is linear on $e(C_\sigma)$.

3. The restriction $v|_{e(C_\sigma)}$ of v onto $e(C_\sigma)$ has an unique linear extension I onto $U = \text{Lin}\,(e(C_\sigma))$. I is a linear order-isomorphism of U onto $K = \text{Lin}\,(v(V))$ with respect to the ordering of U and K induced by ϱ and σ, respectively.

4. $v \cdot e$ is linear.

5. $v \cdot e$ has a unique linear extension $P : H \to H$. We have $0 \leq P \leq I$ with respect to the ordering induced into $\text{Hom}\,(H, H)$ and $P^2 = P$, i. e. P is a *positive linear projection* over H with $P \leq I$.

6. e possesses a linear extension E onto H. E is a *complete pre-Riesz homomorphism* $E : H \to V$, i.e. the following properties (i) and (ii) hold.

(i) E is a *pre-Riesz homomorphism*, i.e. E is linear, isotone and satisfies the 'interpolation property'
$\forall\, x \in V, \gamma, \beta, \sigma \in H : (\sigma \leq \gamma, \beta \;\wedge\; x \leq E(\gamma), E(\beta)) \Rightarrow$
$\exists\, \alpha \in H : \sigma \leq \alpha \leq \gamma, \beta \;\wedge\; x \leq E(\alpha) \leq E(\gamma), E(\beta)$.

(ii) For all downwards directed and bounded subsets A of H and elements $z \in H$ $z = \inf A \Rightarrow E(z) = \inf E(A)$.

7. With the mappings I, P and E introduced in 3., 5. and 6. we have $P = I \cdot E$ and $E = I^{-1} \cdot P$.

⌈ 1.: From theorem 6.8.
2.: $-y, y \leq e \cdot v(y) \leq e \cdot v(x) + e \cdot v(y)$ implies $v(y) \leq v(e \cdot v(x) + e \cdot v(y))$, hence $\varepsilon := v(x) + v(y) - v(e \cdot v(x) + e \cdot v(y)) \leq v(x)$, i.e. $0 \leq v(x) - \varepsilon$. Since v is sublinear and $v \cdot e \cdot v = v$, we have also $0 \leq \varepsilon$. We conclude $v \cdot e(v(x) - \varepsilon) = v(e(v(x)) - e(\varepsilon)) = v(e \cdot v(x) - e \cdot v(x) - e \cdot v(y) + e \cdot v(e \cdot v(x) + e \cdot v(y))) = v(-e \cdot v(y) + e \cdot v \cdot e(v(x) + v(y))) = v(-e \cdot v(y) + e(v(x) + v(y))) = v(e(v(x))) = v(x)$. Therefrom follows $v(x) = v \cdot e(v(x) - \varepsilon) \leq v(x) - \varepsilon$, hence $\varepsilon \leq 0$. Because of $0 \leq \varepsilon$ we have $\varepsilon = 0$, i.e. $v(e \cdot v(x) + e \cdot v(y)) = v(x) + v(y) = v(e \cdot v(x)) + v(e \cdot v(y))$.
3. $v|_{e(C_\sigma)}$ is an order isomorphism (lemma 6.5). Since I is linear and $C_\varrho \cap U = e(C_\sigma)$ (theorem 6.11), it remains to be shown that $I(e(C_\sigma)) = K \cap C_\sigma$. $I(e(C_\sigma)) = v(V) \subset K \cap C_\sigma$ holds generally. Let $\gamma = I(x) - I(y) \in C_\sigma$ with $x, y \in e(C_\sigma)$. Since $I(y) \leq I(x)$, $y \leq x$ because of lemma 6.5 (x). From $C_\varrho \cap U = e(C_\sigma)$ follows $x - y \in e(C_\sigma)$, i.e. $I(x) - I(y) = I(x-y)$ and $\gamma \in I(e(C_\sigma))$.
4. The statement follows from the fact that e is linear and v is linear in $e(C_\sigma)$.
5. Since C_σ is generating, $v \cdot e$ can be uniquely extended onto H. $0 \leq P$ follows from $v \cdot e(C_\sigma) \subset C_\sigma$. $P \leq I$ since $v \cdot e$ is a hull operator and therefore extensive. Likewise $P^2 = P$ follows.
6. (i): Let $\sigma, \gamma, \beta \in H$, $\sigma \leq \gamma$, $\sigma \leq \beta$, $x \leq E(\gamma)$, $x \leq E(\beta)$. Then we have $x - E(\sigma) \leq E(\gamma - \sigma)$, $E(\beta - \sigma)$ and $0 \leq \gamma - \sigma$, $0 \leq \beta - \sigma$. Let $z := E(\gamma - \sigma) \wedge E(\beta - \sigma)$. We obtain $v(z) \leq v \cdot e(\gamma - \sigma) \leq \gamma - \sigma$ and $v(z) \leq v \cdot e(\beta - \sigma) \leq \beta - \sigma$. Let $\alpha := v(z) + \sigma$. We have $\alpha \leq \gamma, \beta$. Since $0 \leq z$ and $z \in U$, $e \cdot v(z) = z$, hence $E(\alpha) = z + E(\sigma)$. Because of the definition of z we also have $x - E(\sigma) \leq z$, therefore $x \leq E(\alpha)$.
(ii): Let $z = \inf A$. We have $0 = \inf (A - z)$. Using lemma 6.9 we obtain $0 = \inf_{C_\varrho} e(A - z)$. Since $U = E(H)$ is a subvector-

lattice, lemma 6.9 can be applied once more in order to obtain $0 = \inf_U e(A - z)$, i.e. $E(z) = \inf_U E(A)$. In an order-projective subvector-lattice, and thereby also in U, each relative infimum is also an infimum in the whole space. Therefore the proof is complete.
7. For all $\gamma \in C_\sigma$ $P(\gamma) = \nu \cdot e(\gamma) = I \cdot E(\gamma)$. But this equality holds even for all $\gamma \in H$, since the linear extension of $\nu \cdot e$ is unique. Since I^{-1} exists, from $P = I \cdot E$ we obtain $E = I^{-1} \cdot P$.⌋

If a mapping $E: H \to V$ is a pre-Riesz homomorphism, then we can prove

(6.14) $\forall\ \alpha, \beta, \gamma \in H,\ x, y \in V$: $(\alpha \leq \beta, \gamma\ \wedge\ x, y \leq E(\beta), E(\gamma)) \Rightarrow \exists\ \delta \in H: \alpha \leq \delta \leq \beta, \gamma\ \wedge\ x, y \leq E(\delta) \leq E(\beta), E(\gamma)$.

⌈Let $\alpha, \beta, \gamma \in H$, $x, y \in V$ such that $\alpha \leq \beta, \gamma$ and $x, y \leq E(\beta), E(\gamma)$. Because of the definition of a pre-Riesz homomorphism there exists a $\tau \in H$ such that $\alpha \leq \tau \leq \beta, \gamma$ and $x \leq E(\tau)$. If we use this defining property once more, then we get a $\delta \in H$ such that $\tau \leq \delta \leq \beta, \gamma$ and $y \leq E(\delta)$. Since E is isotone, $E(\tau) \leq E(\delta)$, hence $x \leq E(\delta)$.⌋

If (H, σ) and (V, ϱ) are vector lattices, then one can easily prove that the pre-Riesz homomorphisms of H into V coincide with the lattice homomorphisms of H into V.

A linear projection P of an ordered vector space (H, σ) with the property $0 \leq P \leq I$ decomposes H into an *order-direct* sum of subspaces K and L, i.e. $H = K + L$, $K \cap L = \{0\}$ and $(K \cap C_\sigma) + (L \cap C_\sigma) = C_\sigma$. K and L are called *order-direct summands* of (H, σ). The order-direct summands are obtained from P by defining $K := P(H)$ and $L := (I - P)(H)$. $H = K + L$ and $K \cap L = \{0\}$ can be easily verified. Let $x \in C_\sigma$. Because of $0 \leq P \leq I$ we have $(I - P)(x) \in C_\sigma$ and $P(x) \in C_\sigma$. Therefore $x \in K \cap C_\sigma + L \cap C_\sigma$. Thereby we obtain the following

(6.15) **Corollary,** Let $\nu : V \to C_\sigma$ and $e : C_\sigma \to C_\varrho$ be mappings such that e is the absolute-inverse of ν and let e be linear. Then we have

1. $K = \operatorname{Lin}(\nu(V))$ is an ideal in (H, σ) and $\nu(V)$ is a *face* of C_σ; in particular, the image set $\nu(V)$ is a convex cone.

2. K is an order-direct summand of H.

3. K is a lattice. With respect to (H, σ) K is a subvector-lattice.

⌈1. $\nu(V) = C_\sigma \cap K$ because of $\nu(V) = \nu \cdot e(C_\sigma) = P(C_\sigma)$ and 2. 2. is true since P is a linear projection satisfying $0 \leq P \leq I$. 3. K is the order isomorphic image of U with respect to the induced orderings. But U is a lattice as stated in corollary 6.12. The remaining steps follow with 2.⌋

An order-interval norm possessing a linear absolute-inverse determines a subvector-lattice in its preimage space (V, ϱ), a subvector-lattice K in its image space (H, σ) and a linear order isomorphism of U onto K. The converse also holds in the following form.

(6.16) **Theorem** Let U be an order-projective subvector-lattice of (V, ϱ), K an order-direct summand of (H, σ) and $I : U \to K$ a linear order isomorphism of U onto K. Then we have

(i) $\forall x \in V : \exists z \in U : z = \inf \{y \in U : -x, x \leq y\}$.

(ii) $\operatorname{abs}_U : V \to V$ with $\operatorname{abs}_U(x) := \inf \{y \in U : -x, x \leq y\}$ is an absolute value operator over V with kernel $U \cap C_\varrho$.

(iii) $\nu := I \cdot \operatorname{abs}_U$ is an order-interval norm.

(iv) On C_σ $E := I^{-1} \cdot P$ is the linear absolute-inverse of ν, where P is defined to be the canonical projection of H onto the direct summand K of H.

⌈(i): Let $\operatorname{lub}(x/U)$ denote $\inf \{z \in U : x \leq z\}$ for $x \in V$. Then we have $z := \operatorname{lub}(x/U) \wedge \operatorname{lub}(-x/U) = \inf \{y \in U : -x, x \leq y\}$.

(ii): The properties symmetric-monotone, symmetric-extensive and idempotent are immediate consequences of the definition of abs_U. Just as easily follows the rest. (iii) and (iv) follow from corollary 6.7 together with theorem 6.13.⌋

The construction of order-interval norms is closely connected with the determination of order-projective subvector-lattices of ordered vector spaces as is expressed by the above theorem. We give a characterization of these subspaces in the case of directed-complete ordered vector spaces (see § 2 for the definition of the latter spaces).

(6.17) **Theorem.** Let (V, ϱ) be a directed-complete ordered vector space. A subspace U of V is an order-projective subvector-lattice of (V, ϱ) if and only if the following three conditions are satisfied.

(i) U is a sublattice.

(ii) U is band-closed, i.e. $\forall\, A \subset U, x \in V: x = \inf A \Rightarrow x \in U$.

(iii) U is cofinal in V, i.e. $\forall\, x \in V: \exists\, y \in U: x \leq y$.

⌈Let $x \in V$. We prove the existence of a $z \in U$ such that $z = \inf \{y \in U: x \leq y\}$. Let $A = \{y \in U: x \leq y\}$. $A \neq \emptyset$. Since U is a sublattice, A is downwards directed and bounded by x. Therefore $z = \inf A$ exists. Since U is band-closed, $z \in U$, hence $z = \inf \{y \in U: x \leq y\} = \mathrm{lub}\,(x/U) \in U$.⌋

Let E be a linear mapping of H into V. If a mapping ν of V into C_σ exists, whose absolute-inverse is given by the restriction of E onto C_σ, then theorem 6.13 statement 6. tells us that E is a complete pre-Riesz homomorphism. The following theorem shows that the converse is true, whenever H is directed-complete.

(6.18) **Theorem.** Let (H, σ) be directed-complete and $H = C_\sigma - C_\sigma$ and let E be a linear mapping of H into V. Then

there exists a mapping $\nu: V \to C_\sigma$ such that E is the absolute-inverse of ν

$\bowtie$ E is a complete pre-Riesz homomorphism and $E(H)$ is cofinal in V.

⌈ $<$: We have to construct a mapping $\nu: V \to H$ such that $\{x \in V: \nu(x) \leq \gamma\} = [-E(\gamma), E(\gamma)]$ for all $\gamma \in C_\sigma$. Let $x \in V$ and $A_x = \{\gamma \in C_\sigma: -x, x \leq E(\gamma)\}$. $A_x \neq \emptyset$ since $E(H)$ is cofinal. Because of property 6.14 A is directed downwards. 0 is a lower bound of A_x, therefore exists a $\gamma \in C_\sigma: \gamma = \inf A_x$. Let $\nu: V \to H$ such that $\forall x \in V\ \nu(x) = \inf\{\gamma \in C_\sigma: -x, x \leq E(\gamma)\}$. $x \in [-E(\gamma), E(\gamma)]$ implies $\nu(x) \leq \gamma$ because of the definition of ν. Conversely, let $\nu(x) \leq \gamma$. Since E preserves infima and, in particular, is isotone, $\inf\{E(\alpha): \alpha \in C_\sigma, -x, x \leq E(\alpha)\} = E(\nu(x))$ and $E(\nu(x)) \leq E(\gamma)$. But this implies $-x, x \leq E(\nu(x)) \leq E(\gamma)$. Thereby we have shown that E is the absolute-inverse of ν.⌋

§ 7. The regularity of (al)-norms and (am)-norms.

The aim of this paragraph is the formulation of necessary and sufficient conditions for the regularity of (al)-norms and (am)-norms. We need a preliminary lemma.

(7.1) **Lemma.** Let B be a linear mapping of V into H and let $0 \leq B$. Then for all downwards directed and bounded subsets A of H^0 and elements γ' of H^0 we have: $\gamma' = \inf_0 A \Rightarrow B^T(\gamma') = \inf_0 B^T(A)$.

⌈ From the remark after 2.3 we get $\forall \gamma \in C_\sigma, x \in C_\varrho$ $\gamma'(\gamma) = \inf\ \{\beta'(\gamma): \beta' \in A\}$ and $(\inf_0 B^T(A))(x) = \inf\{\beta' \cdot B(x): \beta' \in A\}$. Hence $(B^T(\gamma'))(x) = \gamma'(B(x)) = (\inf B^T(A))(x)$.⌋

(7.2) **Theorem.** Let $p: V \to H$ be an (al)-norm and B_p its associated linear mapping. Then

p is regular $\bowtie$ B_p^T is a pre-Riesz homomorphism of H^0 into V^0.

⌈ $>$: From theorem 5.9 together with theorem 6.13 statement 6. (i).

$\lessdot$: Let $f \in V^b$, i.e. $\gamma' \in B_p[f]$ for a $\gamma' \in C^0_\sigma$. From lemma 5.7, $f \leq B^T_p(\gamma')$, i.e. $B^T_p(H^0)$ is cofinal in V^b. Because of 7.1 B^T_p is a complete pre-Riesz homomorphism, therefore, applying theorem 6.18, there exists a mapping $\nu: V^b \to H^0$, such that B^T_p is the absolute-inverse of ν, i.e. such that $K_\nu[\gamma'] = [-B^T_p(\gamma'), B^T_p(\gamma')]$ for all $\gamma' \in C^0_\sigma$. Since also $K^d_p[\gamma'] = [-B^T_p(\gamma'), B^T_p(\gamma')]$ because of 5.7 (1.), $\gamma' \in B_p[f] \Leftrightarrow \nu(f) \leq \gamma'$. That means, because of 2.2, that p is regular and ν is equal to the dual norm of p.⌋

We now characterize the regularity of (am)-norms.

(7.3) **Theorem.** Let $p: V \to H$ be an order-interval norm with a linear associated mapping $E_p: H \to V$. Then

p is regular $\gtrless$ E^T_p has the following *interpolation property*

$\forall f, g, h, k \in V^b: (f, g \leq h, k) \Rightarrow \exists\, l \in V^b: f, g \leq l \wedge E^T_p(l) \leq E^T_p(h), E^T_p(k)$.

⌈ $\gtrdot$: Theorem 5.10 together with Lemma 5.1.
$\lessdot$: Let $f \in V^b$, $\beta', \gamma' \in C^0_\sigma$, $\beta' \in B_p[f]$ and $\gamma' \in B_p[f]$. Because of Lemma 5.7 (2.) there exists a $g \in V^b$ such that $-f, f \leq g$ and $g \cdot E_p = E^T_p(g) \leq \gamma'$. Because of the interpolation property $-f, f \leq l$, $E^T_p(l) \leq E^T_p(h), E^T_p(k)$ with $l \in V^b$ and $E^T_p(l) \in B_p[f]$. It follows that $B_p[f]$ is downwards directed. Now, theorem 2.4 is applied.⌋

(7.4) **Theorem.** Let $p: V \to H$ be an order-interval norm and e_p its absolute-inverse. Let V^b be point-distinguishing for V. Then

p is completely regular $\gtrless$

(i) e_p is linear and

(ii) $\forall x, y \in e(C_\sigma), f \in V^b$: $\sup f([-x, x]) + \sup f([-y, y]) = \sup f([-x-y, x+y])$, i.e. the closed images $f([-x, x])^a$ of the intervals $[-x, x]$ for $x \in e(C_\sigma)$ are additive.

⌈ $>$: $\forall f \in C_\varrho^0 \cap V^b, \gamma \in C_\sigma$ we have $p^d(f)(\gamma) = \sup f(K_p[\gamma]) = \sup f([-e_p(\gamma), e_p(\gamma)]) = f(e_p(\gamma))$. Since $p^d(f)$ is additive, also $f \cdot e_p$ is additive. $f(e_p(\gamma + \beta)) = f(e_p(\gamma)) + f(e_p(\beta)) = f(e_p(\gamma) + e_p(\beta))$. Since, because of 4.6, V^b is an ideal in V^0, $V^b \cap C_\varrho^0$ is point-distinguishing for V. Therefore $e_p(\gamma + \beta) = e_p(\gamma) + e_p(\beta)$. Now let $f \in V^b$. Since again $p^d(f)(\gamma) = \sup f(K_p[\gamma]) = \sup f([-e_p(\gamma), e_p(\gamma)]) \; \forall \gamma \in C_\sigma$, we obtain (ii). $<$: Because $u_f(\gamma) := \sup f(K_p[\gamma]) = \sup f([-e_p(\gamma), e_p(\gamma)])$ is linear, the proof of the statement follows from the definition of the complete regularity.⌋

In a vector lattice the intervals $[x, y]$ are additive, i.e. $[x, y] + [u, v] = [x + u, y + v]$. Therefore the condition (ii) in the preceding theorem is satisfied whenever (V, ϱ) is a lattice.

§ 8. (AL)-norms and (AM)-norms on vector lattices.

Following F. L. Bauer [1] a norm p on a vector lattice (V, ϱ) with values in a vector lattice (H, σ) is an (L)-norm, if

(L) $\quad p(x + y) = p(x) + p(y), \; \forall x, y \in C_\varrho$,

and an (M)-norm, if

(M) $\quad p(x \vee y) = p(x) \vee p(y), \; \forall x, y \in C_\varrho$.

p is called absolute, if $p(x) = p(|x|), \; \forall x \in V$. Correspondingly an absolute (L)-norm and an absolute (M)-norm are called an *(AL)-norm* and an *(AM)-norm*, respectively.

In this last paragraph we investigate (l)-, (m)-, (al)- and (am)-norms p on a vector lattice (V, ϱ) with values in a vector lattice (H, σ). We show among other things that the two last ones coincide with the (AL)- and (AM)-norms, respectively. The duality theorems for (AL)- and (AM)-norms can be obtained as corollaries from the corresponding theorems about (al)- and (am)-norms. A remarkable peculiarity appears for (AL)- and (AM)-norms in so far, as lemma 4.5 can now be proved without using the theorem of Hahn-Banach. That means that all the duality theorems can be proved without using the Hahn-Banach theorem as well. We will give alternative proofs.

Let (V, ϱ) and (H, σ) be vector lattices and p an H-valued norm on V. At first we characterize those monotonicity properties for norms which we have introduced in the §§ 3–4.

(8.1) **Theorem.** Let p be an H-valued norm on V. Then we have

1. p is monotone over $C_\varrho \Leftrightarrow \forall\, x, y \in V$:
   $p(|x|) \vee p(|y|) \leq p(|x| \vee |y|)$.

2. p is symmetric-monotone $\Leftrightarrow \forall\, x, y \in V$:
   $p(x) \vee p(y) \leq p(|x| \vee |y|)$.

3. p is order-convex $\Leftrightarrow \forall\, x, y, z \in V$:
   $p(z) \leq p(z \vee x) \vee p(-z \vee y)$.

⌈1. Clear. 2. $>$: For all $x, y \in V$ we have $-(|x| \vee |y|) \leq x, y \leq |x| \vee |y|$. Therefrom the desired equality follows. $<$: Let $-y \leq x \leq y$. $y = |x| \vee |y|$. Hence $p(x) \leq p(|x| \vee |y|) = p(y)$. 3. $>$: $z \wedge -y \leq z \leq z \vee x$ implies $p(z) \leq p(z \vee x) \vee p(z \wedge -y)$. Thereby, because of $p(z \wedge -y) = p(-z \vee y)$, the inequality is proved. $<$: $x \leq z \leq y$ implies $x = x \wedge z$ and $y = z \vee x$. The rest is clear.⌋

For monotone norms we characterize the directedness properties in the following theorem. To abbreviate, here we call a norm *monotone*, if it is monotone over C_ϱ.

(8.2) **Theorem.** Let p be a monotone norm on V with values in H. Then

1. p is directed $\Leftrightarrow \forall\, x, y \in V: p(|x| \vee |y|) \leq p(x) \vee p(y) \Leftrightarrow$
   p is directed over $C_\varrho \Leftrightarrow \forall\, x, y \in V: p(|x| \vee |y|)$
   $= p(|x|) \vee p(|y|)$.

2. p is symmetric-directed $\Leftrightarrow \forall\, x \in V: p(|x|) \leq p(x)$.

3. p is positive-directed $\Leftrightarrow \forall\, x \in V: p(x^+) \leq p(x)$, where $x^+ := x \vee 0$.

⌈1. We show, if p is directed over C_ϱ, then the first inequality holds. The rest is an easy consequence thereof. Let $x, y \in V$. From $x^+ = \frac{1}{2}(|x| + x)$ and $x^- = \frac{1}{2}(|x| - x)$ we obtain $p(x^+), p(x^-) \leq \frac{1}{2}(p(|x|) + p(x))$. Since p is directed over C_ϱ, $x^+, x^- \leq z \leq p(z) \leq \frac{1}{2}(p(|x|) + p(x))$ for a certain $z \in C_\varrho$. $|x| = x^+ \vee x^- \leq z$ implies $p(|x|) \leq \frac{1}{2}(p(|x|) + p(x))$, because p is monotone. Hence $p(|x|) \leq p(x)$. Similarly, we obtain $p(|y|) \leq p(y)$. Again, since p is directed and monotone over C_ϱ, we have $p(|x| \vee |y|) \leq p(x) \vee p(y)$.
2. and 3. are clear.⌋

(8.3) **Theorem.** Let p be an H-valued norm on V. Then

p is absolute-monotone $\Leftrightarrow$ p is absolute and monotone $\Leftrightarrow$ p is *lattice-monotonic* [13], i.e. $\forall\, x, y \in V: |x| \leq |y| \Rightarrow p(x) \leq p(y)$.

⌈Let p be absolute-monotone, i.e. symmetric-monotone and symmetric-directed. $p(x) \leq p(|x|)$, $\forall\, x \in V$ follows from 8.1.(2.). Moreover, because of 8.2.(2.), we have $p(x) = p(|x|)$, $\forall\, x \in V$, i.e. p is absolute. p is monotone, since p is even symmetric-monotone. Now, let p be absolute and monotone and let $|x| \leq |y|$. $p(|x|) \vee p(|x| \vee |y|) \leq p(|x| \vee |y|) = p(|y|)$ follows from 8.1.(1.). Since p is absolute, we obtain $p(x) \leq p(y)$. If we suppose that p is lattice-monotonic, then $p(|x|) \vee p(|y|) \leq p(|x| \vee |y|)$, $\forall\, x, y \in V$ follows immediately. From $||x|| \leq |x|$ we also obtain $p(|x|) \leq p(x)$, $\forall\, x \in V$, i.e. p is absolute-monotone.⌋

The preceding theorems yield now quite simply that the (al)- and (am)-norms coincide with the (AL)- and (AM)-norms, respectively.

(8.4) **Theorem.** Let p be an H-valued norm on V. Then

1. p is an (al)-norm $\Leftrightarrow$ p is an (AL)-norm $\Leftrightarrow$
   $\forall\, x, y \in V: p(|x| + |y|) = p(x) + p(y)$.

2. p is an (am)-norm $\Leftrightarrow$ p is an (AM)-norm $\Leftrightarrow$
   $\forall\, x, y \in V: p(|x| \vee |y|) = p(x) \vee p(y)$.

⌈1. Let p be an (al)-norm. According to 8.3, p is absolute, hence p is an (AL)-norm. Obviously, the equation is satisfied if p is an (AL)-norm. From $p(|x| + |y|) = p(x) + p(y)$, $\forall\, x, y \in V$ we obtain $p(|x|) = p(x)$, $\forall\, x \in V$. But every (l)-norm is also monotone, i.e., according to 8.3, p is absolute-monotone. 2. Let p be an (am)-norm. 8.3 together with 8.2.(1.) yields that p is an (AM)-norm. Obviously, the equation is satisfied if p is an (AM)-norm. On the other hand from the equation together with 8.1.(2.) and 8.2.(2.) follows that p is absolute-monotone.⌋

Now we turn to the duality theorems. At first we give an alternative proof of lemma 4.5.(3.). Let $p: V \to H$ be a norm and K_p^{d} the dual family of the family K_p of the norm balls of p, i.e. $K_p^{\mathrm{d}}[\gamma'] = \{f \in V^*: f(x) \leq \gamma' \cdot p(x),\ \forall\, x \in V\}$, $\gamma' \in C_\sigma^0$.

(8.5) **Lemma.** Let $\gamma' \in C_\sigma^0$. Then

p is absolute-monotone $\Rightarrow$ $K_p^{\mathrm{d}}[\gamma']$ is absolute-order-convex.

⌈Let $f(x) \leq \gamma' \cdot p(x)$, $\forall\, x \in V$. As is well-known $|f|(|y|) = \sup f([-|y|, |y|])$ for all $y \in V$. Let $z \in V$. Because of $f(x) \leq \gamma' \cdot p(x) \leq \gamma' \cdot p(|z|)$ for all $x \in [-|z|, |z|]$ we have $|f|(z) \leq |f|(|z|) \leq \gamma' \cdot p(|z|) = \gamma' \cdot p(z)$, i.e. $|f| \in K_p^{\mathrm{d}}[\gamma']$. Therefore $K_p^{\mathrm{d}}[\gamma']$ is symmetric-directed. Because of lemma 4.5.(1.) $K_p^{\mathrm{d}}[\gamma']$ is also symmetric-order-convex.⌋

Let $p: V \to H$ be an absolute-monotone norm. One consequence of lemma 8.5 is the fact that the space V^{b} of all bounded linear functionals on V is an ideal in V^0. Therefore V^{b} is a Dedekind-complete vector lattice as well as V^0. The following theorem corresponds to theorem 4.7. For its proof lemma 8.5 is used together with 3.9 and theorem 4.4.(3.).

(8.6) **Theorem.** Let $p : V \to H$ be a regular lattice-monotonic norm. Then the dual norm p^{d} is also lattice-monotonic.

For the proof of the theorems 5.9–5.11 we can also use lemma 8.5 and thereby manage to prove them without using theorem 4.7 or the Hahn-Banach theorem. Moreover, these theorems can be slightly modified by substituting (al)- and (am)- with (AL)- and (AM)- in their text. We do not present the details.

In the following we deal with questions concerning the regularity of (AL)- and (AM)-norms. Since in the case of vector lattices the pre-Riesz homomorphisms coincide with the linear lattice homomorphisms, from theorem 7.2 we obtain the following

(8.7) **Theorem.** Let $p : V \to H$ be an (AL)-norm with the associated linear mapping $B_p : V \to H$. Then

p is regular $\iff$ B_p^{T} is a linear lattice homomorphism.

We have already mentioned that in vector lattices the condition (ii) in theorem 7.4 is always satisfied. Therefore the following theorem is a consequence of 7.4.

(8.8) **Theorem.** Let $p : V \to H$ be an order-interval norm with the associated mapping $e_p : C_\sigma \to C_\varrho$. Let V^{b} be point-distinguishing for V. Then

p is completely-regular $\iff$ e_p is linear.

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